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Stochastic Lagrangian perturbation of Lie transport and applications to fluids

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A B S T R A C T

In this paper, we propose a novel stochastic Lagrangian formulation of dissipatively perturbed Lie transport, which is based on the *statistical generalized Cauchy invariants equation.* This formulation consists of, first, finding a convenient Lagrangian formulation of the Lie transport equation involving particle trajectories, for instance the *backward generalized Cauchy invariants equation* (a Lagrangian formulation for Lie-advected exact p-forms, which is the Hodge dual of a generalization of the Cauchy vorticity formula), and second, performing a stochastic perturbation of the velocity-driven particle trajectories by adding to them white noises. Finally, using Itô's calculus, an ensemble average of the stochastically perturbated generalized Cauchy invariants equation allows us to obtain the Lie transport equation perturbed by a deterministic potentially dissipative term given by the sum of squares of some Lie derivative operators. A remarkable property of this equation is that it satisfies a statistical Kelvin–Helmholtz theorem of conservation of circulation and flux. These results are obtained on periodic Euclidean spaces as well as on smooth closed Riemannian manifolds. In particular, we recover and thus generalize the Constantin–Iyer results on the stochastic Lagrangian formulation of the incompressible Navier–Stokes equations to a larger class of (deterministic) dissipative PDEs. A first application of this stochastic Lagrangian formulation is the derivation of new Lagrangian formulations for non-ideal (dissipative) hydrodynamic and magnetohydrodynamic models on flat and curved spaces, and in particular we obtain stochastic-Lagrangian incompressible extended MHD equations. As a second application, we use this new stochastic Lagrangian formulation to study the local well-posedness, the non-resistive limit and the global existence of classical solutions for the non-ideal incompressible extended MHD on the flat torus. Global-in-time existence is proved for small magnetic Reynolds numbers, that is, either for small initial data or large resistivity.

1. Introduction

Consider a smooth closed Riemannian manifold M of dimension n, and β_t a smooth time-dependent family of exact exterior differential p-forms on M, i.e., such that $\beta_t = d\alpha_t$, where α_t is a $(p-1)$-form and

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d is the exterior derivative. We also consider $m+1$ smooth time-dependent vector fields $\{V_\ell(t)\}_{\ell=0}^m$ on M, satisfying hypotheses (H2) and (H3) below. Our main result says that the p-form β_t satisfies the following deterministic dissipative transport PDE,

$$\partial_t\beta_t(x) + \pounds_{V_0(t,x)}\beta_t(x) - \frac{1}{2}\sum_{\ell=1}^{m}\pounds_{V_\ell(t,x)}\pounds_{V_\ell(t,x)}\beta_t(x) = 0, \quad x \in M, \quad t \in]0,T], \tag{1}$$

with the given initial condition $\beta_0(x) = \beta_{\rm in}(x) = d\alpha_{\rm in}(x)$, if and only if,

$$\beta_t(x) = \mathbb{E}\big[d(\alpha_{i_1\cdots i_{p-1}}(s,\xi_{t,s}(x))) \wedge d\xi_{t,s}^{i_1}(x) \wedge \cdots \wedge d\xi_{t,s}^{i_{p-1}}(x)\big], \tag{2}$$

for all $x \in M$, and for all times s such that $0 \le s \le t \le T$. Here, the first-order differential operator $\pounds_{V_\ell}$ denotes the Lie derivative with respect to the vector field V_ℓ. The symbol $\mathbb{E}$ denotes the expected value with respect to the Wiener measure and the wedge symbol $\wedge$ is the exterior product between exterior differential forms (see, e.g., [3,36] for an introduction to differential geometry). In Eq. (2), the Lagrangian map $x \mapsto \xi_{t,s}(x)$, of local coordinates $\{\xi_{t,s}^i(x)\}_{i=1}^n$, denotes the position on M at time s of a backward stochastic flow ξ, which is at the position $x \in M$ at time t. In Eq. (2), the standard Einstein summation convention is used. The backward stochastic flow $\xi_{t,s}(x)$, which is involved in Eq. (2), satisfies the following backward Stratonovich's stochastic differential equation: for any smooth test function f on M,

$$f(\xi_{t,s}(x)) = f(x) - \int_s^t \mathrm{d}r\, V_0(r)f(\xi_{t,r}(x)) - \sum_{\ell=1}^{m}\int_s^t V_\ell(r)f(\xi_{t,r}(x)) \circ \hat{\mathrm{d}}W^\ell(r), \tag{3}$$

where $\{W^\ell(t)\}_{\ell=1}^m$ are 1-dimensional independent standard Brownian motions. Here, $\circ\,\hat{\mathrm{d}}W(t)$ denotes backward Stratonovich integration with respect to the Brownian motion $W(t)$, which is defined with respect to the standard probability space and filtration (see, e.g., [92,109], and also Notation 1 below).

We call Eq. (2), supplemented with (3), the statistical generalized Cauchy invariants equation, because it corresponds to a statistical version of the deterministic generalized Cauchy invariants equation obtained in [20]. The Cauchy invariants equation traces back to the pioneering work of Cauchy [32]. Indeed, in his famous paper [32], Cauchy used a Lagrangian formulation, called nowadays the Cauchy invariants equation, to integrate the incompressible Euler equations on the 3-dimensional Euclidean (flat) space by establishing the so-called Cauchy vorticity formula, also known as the vorticity transport formula [122]. In [20], the Cauchy invariants equation is generalized to Lie transport of exact p-forms on Riemannian manifolds with applications to some dissipationless hydrodynamic and magnetohydrodynamic models. Moreover, it is shown in [20] that the Cauchy invariants equation and the Cauchy vorticity formula are Hodge duals of each other. Such generalizations lead to novel Lagrangian formulations of some deterministic Hamiltonian flows arising in dissipationless hydrodynamics and magnetohydrodynamics. It should be emphasized that the (deterministic) generalized Cauchy invariants equation turns out to be a powerful formulation for performing mathematical analysis of some ideal fluid models (with constructive existence proofs) [16,17,19,70,172], and also for designing very accurate numerical schemes of arbitrary order in computational ideal fluid dynamics [86,140].

In this paper, we aim at extending the Lagrangian formulation given by the generalized Cauchy invariants equation for dissipationless Hamiltonian flows to dissipative flows. In doing so, we obtain the same generalized Cauchy invariants equation (as for Hamiltonian Lie transport) but at the price of a statistical average over an ensemble of random realizations, namely the statistical generalized Cauchy invariants Eq. (2), while the associated deterministic dissipative PDE is given by the transport Eq. (1). Then, the system (2)–(3) constitutes a (stochastic) Lagrangian formulation of the dissipative transport PDE (1). Moreover, from this stochastic Lagrangian formulation, we show that such a dissipative equation satisfies a statistical Kelvin–Helmholtz theorem of conservation of circulation and flux.

Choosing the vector fields $\{V_\ell\}_{\ell=1}^m$ such that the system $\{V_\ell, W^\ell\}_{\ell=1}^m$ constitutes a gradient Brownian system (see Section 3.1), the sum of squares of Lie derivative in (1) becomes equivalent to the Hodge–De Rham Laplacian operator. Therefore, we derive novel (stochastic) Lagrangian formulations of some (deterministic) diffusive hydrodynamic and magnetohydrodynamic equations on Euclidean and curved spaces (see Section 3). In particular, we recover the stochastic Lagrangian representation of Constantin–Iyer [37,38] for the incompressible Navier–Stokes equations on $\mathbb{R}^3$ (see Section 3.2). Actually our result can be seen as a generalization of the Constantin–Iyer result [37,38] (written in terms of vorticity) to the dissipative transport Eq. (1) posed on Riemannian manifolds. A different extension (written in terms of velocity and using the Weber's representation formula) to manifolds of the Constantin–Iyer stochastic formulation [37] of the incompressible Navier–Stokes equations can be found in [60]. We should also mention some connected results. The first one is the stochastic mean-field derivation of the incompressible Navier–Stokes equations by infinite-dimensional mean-field stochastic differential equation [87]. The other ones are the works [10,56] (see also [145] for related results), which show a link between the Brenier's generalized flow type solutions for the incompressible Euler equation on Euclidean space [56] and manifolds [10], and the solutions of the incompressible Navier–Stokes equations on Euclidean space and manifolds [47]. We also recover some of the results of [55], where the author applies the stochastic Lagrangian framework of Constantin–Iyer [37] to some magnetohydrodynamic models on $\mathbb{R}^3$. Moreover, in [55] the author uses the Kuznetsov–Ruban's Lagrangian representation of vorticity and magnetic field vectors on $\mathbb{R}^3$ [110,149] to obtain a stochastic Lagrangian formulation of the standard incompressible MHD equations. We can also mention the work [165] in which the Constantin–Iyer framework [37,94,95] is used to deal with the damped Navier–Stokes equations and the Boussinesq system. It is worthwhile to note that in the above works [10,55,56,60,87,145,165] the authors use forward-in-time stochastic formulations, whereas our results rely crucially on backward-in-time stochastic formulations. Indeed, as observed in [38] for the incompressible Navier–Stokes equations, the convenient formulation to take into account boundary conditions is the backward-in-time formulation, which also does not require computation of the spatial inverse of the stochastic flow as it is required for the forward-in-time stochastic formulation. As for backward semi-Lagrangian schemes [18,21,83,112,166] designed to solve numerically the advection equation, the drawback of a backward-in-time stochastic differential equation is that it makes the formulation more implicit since one part of the information is known at the final time (the final position of the stochastic flow ξ) and the other part at the initial time (the initial vector fields $\{V_\ell(t=0)\}_{\ell=0}^m$ and the initial p-form β_0). Of course, such a drawback arises when the vector fields $\{V_\ell\}_{\ell=0}^m$ depend on the p-form β through some couplings, i.e., generally for nonlinear equations. When the vector fields $\{V_\ell\}_{\ell=0}^m$ are given "external" vector fields, such a drawback disappears and Eqs. (1) or system (2)–(3) constitutes a linear problem. By contrast, in a forward-in-time stochastic formulation, the knowledge of all initial data (initial position of the stochastic flow ξ, the initial vector fields $\{V_\ell(t=0)\}_{\ell=0}^m$ and the initial p-form β_0) is sufficient, but the spatial inverse of the stochastic flow is however needed, as observed in the Weber's formulation of the velocity vector satisfying the incompressible Navier–Stokes equations [38].

We think that such stochastic Lagrangian formulations of deterministic Eulerian dissipative transport equations may be useful from several points of view. First, it offers an alternative and self-consistent framework to perform the mathematical analysis of some deterministic dissipative PDEs on Euclidean spaces and on manifolds. This is what we do in Section 4, where the local well-posedness, the non-resistive limit and the global existence of classical solutions for the stochastic-Lagrangian incompressible extended MHD equations are studied on the 3-dimensional flat torus. To clarify and shorten functional estimates, we have only considered the 3-dimensional flat torus, but there is a priori no conceptual difficulties to extend such analysis to a smooth closed manifold, provided that the vectors fields $\{V_\ell\}_{\ell=0}^m$ are smooth enough in space. Second, this stochastic Lagrangian framework is very well suited to extend on manifolds large deviation principles of Freidlin–Wentzell's type obtained in Euclidean spaces [15,69,144]. Third, this (stochastic) Lagrangian framework should be very useful to develop new numerical schemes and to understand some physical mechanisms underlying these models. For example, the authors of [57,58,161,162] use the stochastic Lagrangian representation of Constantin–Iyer [37,38] for the incompressible Navier–Stokes equations to understand production of vorticity and Lagrangian chaos in transitional and turbulent wall-bounded flows (see also [59] for high-conductivity magnetohydrodynamic turbulence). Moreover, the original semi-Lagrangian numerical schemes developed in [86,140] could probably be extended to such stochastic Lagrangian framework, first on Euclidean spaces, then on a sphere for various applications such as numerical weather prediction [83,112,142,146–148,150,166]. In addition, many hydrodynamic and magnetohydrodynamic flows of practical interest evolve on curved spaces, such as geodynamo (e.g. dynamics of the geomagnetic field around a planet) and geophysical flows (e.g. flows in the atmosphere and oceans of planets, climate modeling) [44,77,100,127,131,132,138,171]. To simplify the exposition and the analysis, here, we consider periodic Euclidean spaces and (boundaryless) closed manifolds, but similar results should hold for bounded domains and manifolds with boundaries. This will be the subject of future work.

Finally, we mention some works which are closely related to our results. The first one consists in some stochastic PDEs for modeling stochastic fluid dynamics [4,40,45,46,88,114]. Especially in [4,46] the authors derive on Euclidean spaces some *Lagrangian-averaged* fluid equations involving double Lie-derivative terms by taking the expectation of some stochastic Lagrangian-averaged equations. However, we emphasize that our results concern stochastic Lagrangian formulation of deterministic (and not stochastic) dissipative PDEs. Moreover, the statistical generalized Cauchy invariants Eq. (2) does not appear in the works cited above. Another work [143] seems more closely connected to our results, but unfortunately it is very difficult to read and it remains obscure to the present author because, among other things, it uses complicated tools such as the Riemann–Cartan–Weyl geometry and some theorems and their proofs are elusive.

The outline of the paper is as follows. Section 2 contains our main results about the statistical generalized Cauchy invariants equation (Theorem 1 of Section 2.1) and the statistical Kelvin–Helmholtz invariants (Corollary 1 of Section 2.1), which are followed by their proofs (see Section 2.2). In Section 3, we apply our general results to some non-ideal hydrodynamic and magnetohydrodynamic equations to obtain new stochastic Lagrangian formulations, in a periodic box and on a smooth closed Riemannian manifold, of the incompressible Navier–Stokes equations, of the non-ideal incompressible extended MHD equations, and of the non-ideal two-fluid equations (containing also the inertial electron-MHD). In Section 4, following the spirit of [94,95,168], we perform the mathematical analysis in Hölder spaces of the stochastic-Lagrangian incompressible extended MHD system, by studying, on the 3-dimensional flat torus, its local well-posedness (Theorem 3 of Section 4.1), its non-resistive limit (Theorem 4 of Section 4.2) and its global-in-time classical solutions for small magnetic Reynolds numbers, that is, either for small initial data or large resistivity (Theorem 5 of Section 4.3).

2. A general framework

2.1. Main results

Before stating the main result of this section, we first need to recall an important result concerning the existence, uniqueness and regularity of stochastic flow of diffeomorphisms that are solutions of Stratonovich's stochastic differential equations. For this, on the one hand some notation of differential geometry and stochastic integration must be introduced, and on the other hand some assumptions on the manifold M and the vector fields $\{V_\ell\}_{\ell=0}^m$, are required. Throughout the paper, we use the Einstein summation convention for the parts concerning manifolds.

We start with the following hypothesis, which precises the regularity of the manifold M.

Hypothesis 1. (H1) Let M be a connected closed (compact) $\mathscr{C}^{\mathfrak{p}}$-manifold of dimension n with $\mathfrak{p}$ large enough. Let (M, g, ∇) be a Riemannian manifold, where $g \in \mathscr{C}^{\mathfrak{p}-1}(M)$ is the 2-covariant metric tensor and ∇ is the Riemann–Levi–Civita connection defined on M (see, e.g., [12,43]).

Before setting the next hypotheses, we briefly introduce standard notation for some differential geometry elements, whose precise definitions can be found in the following classical textbooks [3,12,13,22,36,43,96]. Moreover, Appendix B of [20] can serve as a short reminder of the differential geometry notions used here. We also introduce standard notation for stochastic integration, for which we refer the reader to the following textbooks [91,92,108,109,157].

Notation 1.

- *With $k \in \mathbb{N}$ and $0 < \gamma < 1$, we denote the Hölder space $\mathscr{C}^{k,\gamma}(M)$ as the space of all functions on M whose any derivatives up to the order k are γ-Hölder continuous (see, e.g., [12,76,85,159,160]).*
- *Let T_xM be the tangent space to M at x, which is the vector space of tangent vectors to M at $x \in M$.*
- *Let T_x^*M be the cotangent space to M at x, which is the vector space of cotangent vectors to M at $x \in M$, or in other words the set of 1-forms, i.e., the set of linear forms acting on the vectors of T_xM.*
- *Let $\mathfrak{X}^k(M)$ be the space of $\mathscr{C}^k$-vector field on M, i.e., the set of first-order linear differential operators with $\mathscr{C}^k$-coefficients, $k \geq 0$. In other words, if X is a $\mathscr{C}^k$-vector field on M, it can be represented as*
$$X = X(x) = X^i(x)\frac{\partial}{\partial x^i}_{|x} = X^i(x)\frac{\partial}{\partial x^i},$$
where the local vector basis $\{\partial/\partial x^i\}_{i=1}^n$, evaluated at the point $x \in M$, spans the tangent space T_xM and the components $\{X^i(x)\}_{i=1}^n$ of X in this basis are $\mathscr{C}^k$ functions on M.
- *Let $\mathscr{C}^k\Lambda^p(M)$ be the space of differential p-forms on M, which are k-times continuously differentiable on M, with $k \geq 0$.*
- *Using the 2-covariant metric tensor (or 2-forms) g, each tangent space T_xM is endowed with a Riemannian metric given by the inner or scalar product $\langle X, Y\rangle_{T_xM} := g(X,Y) = g_{ij}(x)X^i(x)Y^j(x)$, $\forall\, X,\, Y \in T_xM$, which leads to the natural Riemannian norm $|\cdot|_{T_xM} := \sqrt{\langle\cdot,\cdot\rangle_{T_xM}}$ on T_xM. The Riemannian metric induces an isomorphism between the tangent space T_xM and its dual, the cotangent space T_x^*M. In particular, it induces the following isomorphism, called the lowering (or flat) operator $(\cdot)^\flat : \mathfrak{X}^k(M) \to \mathscr{C}^k\Lambda^1(M)$, and its inverse, the raising (or sharp) operator $(\cdot)^\sharp : \mathscr{C}^k\Lambda^1(M) \to \mathfrak{X}^k(M)$. Then, each cotangent space T_x^*M is also endowed with a Riemannian metric given by the inner product $\langle \alpha, \beta\rangle_{T_x^*M} := g^{ij}(x)\alpha_i(x)\beta_j(x)$, $\forall\, \alpha,\, \beta \in T_x^*M$, which leads to the natural norm $|\cdot|_{T_x^*M} := \sqrt{\langle\cdot,\cdot\rangle_{T_x^*M}}$ on T_x^*M. We denote by $\langle\cdot,\cdot\rangle_{\mathbb{R}^n}$ the canonical inner product in $\mathbb{R}^n$.*
- *Let $d : \mathscr{C}^k\Lambda^p(M) \to \mathscr{C}^{k-1}\Lambda^{p+1}(M)$ be the exterior derivative, and $d^* : \mathscr{C}^k\Lambda^p(M) \to \mathscr{C}^{k-1}\Lambda^{p-1}(M)$ be the exterior coderivative. Using the Hodge star isomorphism $* : \mathscr{C}^k\Lambda^p(M) \to \mathscr{C}^k\Lambda^{d-p}(M)$, the codifferential operator d^* is defined as follows, $d^*\alpha = (-1)^{n(p-1)+1} * d * \alpha$, $\forall\, \alpha \in \mathscr{C}^k\Lambda^p(M)$. The Hodge–De Rham Laplacian is defined by $\Delta_{\mathrm{H}} := -(dd^* + d^*d)$. We recall the Cartan formula, which gives the following alternative definition of the Lie derivative, $\pounds_X = \mathrm{i}_X d + d\mathrm{i}_X$, where $\mathrm{i}_X : \mathscr{C}^k\Lambda^p(M) \to \mathscr{C}^k\Lambda^{p-1}(M)$ is the interior product of a p-form with the vector field $X \in \mathfrak{X}^k$.*
- *Let $X(t) \in \mathscr{C}([0,T]; \mathfrak{X}^k(M))$ be a time-dependent vector field on M, with $k \geq 0$. Let $f \in \mathscr{C}^1(M;\mathbb{R})$ and $\varphi \in \mathscr{C}(M;M)$ be respectively a continuously differentiable real-valued function on M and a continuous map from M to M. The notation $X(t)f(\varphi(x))$ means that the vector field $X(t)$ first acts (as a first-order differential operator) on the function f and then the result is evaluated at the point $\varphi(x)$, i.e.,*
$$X(t)f(\varphi(x)) = X^i(t,\varphi(x))\frac{\partial f}{\partial x^i}(\varphi(x)).$$

- *Throughout the paper, we assume that we work with a stochastic basis* $\big(\Omega,\, \mathcal{F},\, \mathbb{F},\, P,\, \big\{(W^\ell(t))_{t\in[0,T]}\big\}_{\ell=1}^m\big)$ *satisfying the usual conditions of continuity and completeness, that is, a filtered probability space* $(\Omega, \mathcal{F}, \mathbb{F}, P)$ *together with a family* $\big\{(W^\ell(t))_{t\in[0,T]}\big\}_{\ell=1}^m$ *of independent* $\mathbb{R}$*-valued standard Wiener processes (or Brownian motions) that is adapted to the filtration (with two parameters)* $\mathbb{F} = \{\mathcal{F}_{s,t}\}_{0\le s\le t\le T} \subset \mathcal{F}$*, i.e., a complete* σ*-algebra generated by all the increments* $W(\theta) - W(\tau)$, $s \le \theta \le \tau \le t$*, for* $0 \le s < t \le T$.
- *The stochastic differential* $\circ\,\mathrm{d}W(t)$ *(resp.* $\circ\,\hat{\mathrm{d}}W(t)$*) denotes forward (resp. backward) Stratonovich's integration with respect to the Brownian motion* $W(t)$*, while the stochastic differential* $\mathrm{d}W(t)$ *(resp.* $\hat{\mathrm{d}}W(t)$*) denotes forward (resp. backward) Itô's integration with respect to the Brownian motion* $W(t)$.

We continue with the following hypothesis, which fixes the regularity of the vector fields $\{V_\ell\}_{\ell=0}^m$.

Hypothesis 2. (H2) For $k \ge 1$ and $0 < \gamma < 1$, we assume that

(i) $V_0 \in \mathscr{C}([0,T]; \mathfrak{X}^k(M))$, and its components $V_0^i \in L^\infty([0,T]; \mathscr{C}^{k,\gamma}(M))$, for $i \in \{1, \dots, n\}$.
(ii) $V_\ell \in \mathscr{C}([0,T]; \mathfrak{X}^{k+1}(M))$, and its components $V_\ell^i \in L^\infty([0,T]; \mathscr{C}^{k+1,\gamma}(M))$, for $i \in \{1, \dots, n\}$, and $\ell \in \{1, \dots, m\}$.

The last hypothesis we will introduce deals with a coercivity assumption for the following linear differential operator,

$$L := \tfrac{1}{2} \sum_{\ell=1}^m \pounds_{V_\ell} \pounds_{V_\ell} - \pounds_{V_0}, \tag{4}$$

so that L be uniformly elliptic on M. With $\mathcal{L}(\mathbb{R}^m, T_xM)$ denoting the set of all linear mappings from $\mathbb{R}^m$ to T_xM, we define the linear map $\mathcal{P}(x) \in \mathcal{L}(\mathbb{R}^m, T_xM)$ by

$$\mathcal{P}(x)X := \sum_{\ell=1}^m X_\ell V_\ell(x) = \sum_{\ell=1}^m X_\ell V_\ell^i(x) \frac{\partial}{\partial x^i}, \quad \forall\, X \in \mathbb{R}^m, \quad \forall\, x \in M,$$

which can also be represented by a matrix with entries $\mathcal{P}_{ij}(x) = V_j^i(x)$, $i = 1, \dots, n$, and $j = 1, \dots, m$. We recall that in the above expression we have used the Einstein summation convention. The adjoint of $\mathcal{P}(x)$, i.e., the linear map $\mathcal{P}^*(x) \in \mathcal{L}(T_x^*M, \mathbb{R}^m)$, is defined through duality by the standard relation: $(\mathcal{P}^*(x)\alpha)(X) = \alpha(\mathcal{P}(x)X)$, for all $\alpha \in T_x^*M$ and for all $X \in \mathbb{R}^m$. Therefore, we obtain $(\mathcal{P}^*(x)\alpha)(X) = \sum_{\ell=1}^m \alpha_i X_\ell V_\ell^i(x)$. Let $\sigma(L; x) : T_x^*M \to T_xM$ be the symbol of the linear differential operator L defined by (4). Then, we obtain $\sigma(L, x) = \mathcal{P}(x)\mathcal{P}^*(x)$. For all $\alpha \in T_x^*M$, we set $\sigma_\alpha(L; x) = \sigma(L; x)(\alpha) \in T_xM$, i.e.,

$$\sigma_\alpha(L; x) = \mathcal{P}(x)\mathcal{P}^*(x)\alpha = \sum_{\ell=1}^m \alpha_i V_\ell^i(x) V_\ell^j(x) \frac{\partial}{\partial x^j}, \quad \forall\, \alpha \in T_x^*M.$$

We then write $\sigma(L; x)(\alpha, \beta) = \beta(\sigma_\alpha(L; x))$, for all $\beta \in T_x^*M$ and so we consider the symbol $\sigma(L; x)$ as a bilinear form on T_x^*M. Therefore, we obtain

$$\sigma(L; x)(\alpha, \beta) = \sum_{\ell=1}^m \alpha_i V_\ell^i(x) V_\ell^j(x) \beta_j, \quad \forall\, \alpha,\, \beta \in T_x^*M. \tag{5}$$

Since for all $x \in M$, $\mathcal{P}(x)\mathcal{P}^*(x)$ is positive semi-definite, the operator L is semi-elliptic, possibly degenerate, i.e., $\sigma(L; x)(\alpha, \beta) \ge 0$, for all $\alpha,\, \beta \in T_x^*M \backslash \{0\}$, and for all $x \in M$. The operator L is elliptic if and only if $\sigma(L; x)(\alpha, \beta) > 0$, for all $\alpha,\, \beta \in T_x^*M \backslash \{0\}$, and for all $x \in M$. Then, the operator L is elliptic if and only if, for all $x \in M$, $\mathcal{P}(x)\mathcal{P}^*(x)$ is positive definite. For this, it is necessary and sufficient that the linear map $\mathcal{P}(x)$ be surjective for all $x \in M$, or equivalently with a constant maximal rank. Then the symbol $\sigma(L; x)$, which is quadratic, is non-degenerate and so determines a metric on T_xM. Indeed, using the inner product $\langle\cdot,\cdot\rangle_{T_xM}$ on T_xM, a Riemannian metric is induced on M and given by $\langle\mathcal{P}(x)e_1,\mathcal{P}(x)e_2\rangle_{T_xM} = \langle e_1,e_2\rangle_{\mathbb{R}^m}$, for all $e_1, e_2 \in \mathbb{R}^m \cap (\ker\mathcal{P}(x))^\perp$ (the orthogonal of the kernel of $\mathcal{P}(x)$ in $\mathbb{R}^m$). Uniform ellipticity of the operator L is then stated in the following hypothesis.

Hypothesis 3. (H3) The linear differential operator L defined by (4) is uniformly elliptic on M, i.e., there exists a constant $\kappa > 0$ such that

$$\sigma(L;x)(\alpha,\alpha) \geq \kappa|\alpha|^2_{T_x^*M}, \quad \forall\,\alpha \in T_x^*M\backslash\{0\}, \quad \forall\,x \in M.$$

Using Notation 1, the following proposition states the existence, uniqueness and regularity of stochastic flow of diffeomorphisms that are solutions of Stratonovich's stochastic differential equations.

Proposition 1 (*Stochastic Flow of Diffeomorphisms, Kunita [108,109]*)**.** *Let M be a Riemannian manifold satisfying hypothesis* (H1)*. Let $F(t)$ be a continuous-in-time stochastic process taking values in $\mathfrak{X}^k(M)$, with $k \geq 1$, and defined by*

$$F(t) = F(t,x) := tV_0(t,x) + \sum_{\ell=1}^{m} W^\ell(t)V_\ell(t,x), \tag{6}$$

where the vector fields $\{V_\ell\}_{\ell=0}^m$ satisfy hypothesis (H2)*, and the stochastic processes $\{W^\ell(t)\}_{\ell=1}^m$ are standard $\mathbb{R}$-valued $\mathbb{F}$-adapted Brownian motions. Then*

1. *$F(t)$ takes values in strictly complete vector fields on M both to the forward and backward direction.*
2. *$\forall\, f \in \mathscr{C}^q(M)$, with $q \geq 3$, the solution $\xi_{s,t}$ of the following forward Stratonovich's stochastic differential equation based on $F(t)$:*

$$\begin{aligned} f(\xi_{s,t}(x)) &= f(x) + \int_s^t F(\circ\,\mathrm{d}r)f(\xi_{s,r}(x)) \\ &= f(x) + \int_s^t \mathrm{d}r\, V_0(r)f(\xi_{s,r}(x)) + \sum_{\ell=1}^{m}\int_s^t V_\ell(r)f(\xi_{s,r}(x))\circ \mathrm{d}W^\ell(r), \end{aligned} \tag{7}$$

exists, is unique, and defines a continuous-in-time stochastic flow of $\mathscr{C}^{k,\mu}$-diffeomorphisms with $\mu < \gamma$. Moreover, $\xi_{s,t}$ has a unique inverse $\xi_{t,s} = \xi_{s,t}^{-1}$, which satisfies the following backward Stratonovich's stochastic differential equation: $\forall\, f \in \mathscr{C}^q(M)$, with $q \geq 3$,

$$\begin{aligned} f(\xi_{t,s}(x)) &= f(x) - \int_s^t F(\circ\,\hat{\mathrm{d}}r)f(\xi_{t,r}(x)) \\ &= f(x) - \int_s^t \mathrm{d}r\, V_0(r)f(\xi_{t,r}(x)) - \sum_{\ell=1}^{m}\int_s^t V_\ell(r)f(\xi_{t,r}(x))\circ \hat{\mathrm{d}}W^\ell(r). \end{aligned} \tag{8}$$

Proof. Combine Theorem 4.8.5 and Corollary 4.8.6 of [109] or Theorem II.9.2 and Corollary II.9.3 of [108] (see also [92,107]). $\square$

Remark 1. We can express (7) or (8) by using local coordinates. Let $(x^1,\ldots,x^n)$ be a local coordinate in a neighborhood $U \subset M$, where U is a local open subset of M. The coordinate functions $f^i(x) = x^i$, with $i \in \{1,\ldots,n\}$, can serve as a natural set of test functions f in (7) and (8). Therefore, if the flow $\xi_{t,s}(x)$

satisfies (8), then $\xi^i_{t,s}(x) = f^i(\xi_{t,s}(x))$ satisfies

$$\begin{aligned}\xi^i_{t,s}(x) &= x^i - \int_s^t F^i\big(\circ\hat{\mathrm{d}}r,\, \xi_{t,r}(x)\big)\\ &= x^i - \int_s^t \mathrm{d}r\, V_0^i(r, \xi_{t,r}(x)) - \sum_{\ell=1}^m \int_s^t V_\ell^i(r, \xi_{t,r}(x)) \circ \hat{\mathrm{d}}W^\ell(r).\end{aligned}$$

Other interesting sets of test functions f are a local chart (U, ϕ), where the map ϕ is a smooth local bijection from U to a local open subset of $\mathbb{R}^n$, or a smooth isometric embedding $(M, \mathfrak{e})$, where the map $\mathfrak{e}$ is a smooth isometric embedding of M into $\mathbb{R}^m$, with $m \geq n$ (see Section 3.1).

We are now in position to state the main result of this section.

Theorem 1 (*The Statistical Generalized Cauchy Invariants Equation*)**.** *Let M be a Riemannian manifold satisfying hypothesis* (H1)*. Let $\{V_\ell\}_{\ell=0}^m$ be vector fields on M satisfying hypothesis* (H2) *with $k \geq 3$, and hypothesis* (H3)*. We denote by $\xi_{s,t}(x)$ the solution of the forward Stratonovich's stochastic differential Eq.* (7)*, and by $\xi_{t,s} = \xi_{s,t}^{-1}$ its inverse, which is the solution of the backward Stratonovich's stochastic differential Eq.* (8)*. We recall that by Proposition 1, the flows $\xi_{s,t}$ and $\xi_{t,s}$ exist, are unique and are continuous-in-time stochastic flows of $\mathscr{C}^{k,\mu}$-diffeomorphisms on M with $0 < \mu < \gamma$. Let $\beta_{\mathrm{in}} = d\alpha_{\mathrm{in}} \in \mathscr{C}^{2,\mu}\Lambda^p(M)$ be any given exact p-form, with $\alpha_{\mathrm{in}} \in \mathscr{C}^{3,\mu}\Lambda^{p-1}(M)$. Let $\beta \in L^\infty([0,T]; \mathscr{C}^{2,\mu}\Lambda^p(M)) \cap \mathrm{Lip}([0,T]; \mathscr{C}^{0,\mu}\Lambda^p(M))$ be a time-dependent family of exact p-forms, i.e., $\beta = d\alpha$ with $\alpha \in L^\infty([0,T]; \mathscr{C}^{3,\mu}\Lambda^{p-1}(M)) \cap \mathrm{Lip}([0,T]; \mathscr{C}^{1,\mu}\Lambda^{p-1}(M))$.*

Then, β satisfies the following dissipative transport equation,

$$\partial_t \beta_t(x) + \mathcal{L}_{V_0(t,x)}\beta_t(x) - \tfrac{1}{2}\sum_{\ell=1}^m \mathcal{L}_{V_\ell(t,x)}\mathcal{L}_{V_\ell(t,x)}\beta_t(x) = 0, \quad x \in M, \quad t \in]0,T], \tag{9}$$

with $\beta_0(x) = \beta_{\mathrm{in}}(x)$ as initial data, if and only if, for all $x \in M$, and for all s such that $0 \leq s \leq t \leq T$, the following statistical generalized Cauchy invariants equation,

$$\beta_t(x) = \mathbb{E}\big[d(\alpha_{i_1\ldots i_{p-1}}(s) \circ \xi_{t,s}(x)) \wedge d\xi^{i_1}_{t,s}(x) \wedge \cdots \wedge d\xi^{i_{p-1}}_{t,s}(x)\big], \tag{10}$$

is satisfied.

A few remarks are now in order.

Remark 2. Representation formula (10) is named the statistical generalized Cauchy invariants equation because this is a statistical version of the deterministic backward generalized Cauchy invariants Eq. (14) of Theorem 2 below.

Remark 3 (*Boundary Conditions*)**.** In order to not deal with boundary conditions and make the result of Theorem 1 clearer, we consider a closed manifold, i.e., a manifold without boundary. Of course, with suitable boundary conditions, Theorem 1 could certainly be extended to compact manifolds with boundaries. This will be the matter of future work.

Remark 4 (*Well-Posedness*)**.** Under assumptions (H1)–(H3), well-posedness (existence, uniqueness, regularity) of Eq. (9) with initial data in Hölder spaces, is ensured by the standard theory of linear parabolic PDEs with variables coefficients, which are bounded in time and Hölder continuous with respect to the space variables (see, e.g., [82,90,101–105,111,116,120]). Of course, from Theorem 1, representation formula (10) together with well-posedness of the backward Stratonovich's stochastic differential Eq. (8), given by Proposition 1, constitute an (stochastic) alternative proof of well-posedness of PDE (9).

Remark 5 (*Ellipticity Hypothesis*)**.** In fact, uniform ellipticity hypothesis (H3) is rather strong and it can probably be relaxed to the ellipticity hypothesis, i.e., the matrix with entries $(\sum_{\ell=1}^{m} V_\ell^i(x)V_\ell^j(x))_{ij}$ is positive definite for all $x \in M$. In such a case we say that the diffusion (or the operator L defined by (4)) is non-degenerate [48,92]. Furthermore, ellipticity hypothesis can also be weakened to hypoellipticity hypothesis (i.e., roughly speaking, a current β satisfying $L\beta$ smooth implies β smooth), which is ensured by the *Hörmander condition* for hypoellipticity, given in Theorem 1.1 of [89] (see also Theorem 22.2.1 in §22.2 of [90]): the vector fields $\{V_\ell(x)\}_{\ell=1}^m$ together with their iterated Lie brackets (called the Lie algebra generated by $\{V_\ell(x)\}_{\ell=1}^m$) span the tangent space T_xM for each $x \in M$. In such a case, the symbol $\sigma(L;x)$ of the operator L, defined by (5), is positive semi-definite (see Corollary 2.2 of [89]) and the diffusion (or the operator L) is possibly degenerate. From Corollary 4.8.6 of [109] (and thus by Proposition 1 above, see also, Section III.5 of [108]), since $F(t)$ is a strictly complete vector fields on a smooth compact manifold M, both to the forward and backward direction, it means that L is hypoelliptic.

Remark 6 (*Well-Posedness and Critical Regularity*)**.** The regularity assumptions on the vector fields $\{V_\ell\}_{\ell=0}^m$, and in particular for the drift vector field V_0, in Proposition 1 and in Theorem 1 are not optimal and can be weakened. Inspired by the works [6,42] on deterministic ODEs and linear transport theory on Euclidean spaces with rough vector fields, in recent years, much attention has been paid to understand the well-posedness of linear stochastic transport equations (advection equations, continuity equation) on Euclidean spaces with rough vector fields because adding a noise term has the effect to regularize and restore the uniqueness of solutions without noise [11,31,34,61–64,66,67,75,115,119,124,129,136,137,141,167, 169,170]. Therefore, after taking the expectation, the resulting linear deterministic PDEs can inherit such properties. Such phenomenon probably holds also for linear stochastic transport on manifolds, but very few results are established [72,73] (see also [115] for linear stochastic transport of differential k-forms on Euclidean spaces). Our present framework can probably be used to obtain some results of "well-posedness/regularization by noise" type. This should be investigated in future work. However, this phenomenon of well-posedness/regularization by noise seems lost for nonlinear equations since for the simplest nonlinear inviscid Burgers' equation adding noise does not improve even well-posedness [5,65].

From Theorem 1 we can infer the following corollary.

Corollary 1 (*The Statistical Kelvin Circulation Theorem and the Statistical Helmholtz Flux Conservation Theorem*)**.** *Under the same hypotheses as in Theorem 1, for any p-chain c of M we obtain,*

$$\oint_{\partial c} \alpha_t = \mathbb{E}\Big[\oint_{\xi_{t,s}(\partial c)} \alpha_s\Big], \qquad \forall\, s \;\; s.t.\;\; 0 \le s \le t \le T, \tag{11}$$

$$\int_{c} \beta_t = \mathbb{E}\Big[\int_{\xi_{t,s}(c)} \beta_s\Big], \qquad \forall\, s \;\; s.t.\;\; 0 \le s \le t \le T. \tag{12}$$

The proof of Theorem 1 uses a Lagrangian formulation of the Lie transport, which is given by the following theorem. This theorem is the version in the backward direction of Theorem 1 of [20], which is stated in the forward direction.

Theorem 2 (*The Backward Generalized Cauchy Invariants Equation*)**.** *Let M be a Riemannian manifold satisfying hypothesis* (H1). *Let V_0 be a vector field on M, which satisfies hypothesis* (H2) *with $k \ge 1$. We denote by $\xi_{s,t}(x)$ the solution of the forward ordinary differential equation:*

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\xi_{s,t}(x) = V_0(t, \xi_{s,t}(x)), \quad 0 \le s \le t \le T, \\ \xi_{s,s}(x) = x \in M, \end{cases}$$

and by $\xi_{t,s} = \xi_{s,t}^{-1}$ its inverse, which is the solution of the same, but backward, ordinary differential equation:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}s}\xi_{t,s}(x) = V_0(s, \xi_{t,s}(x)), \quad 0 \le s \le t \le T, \\ \xi_{t,t}(x) = x \in M. \end{cases}$$

We recall that from Cauchy–Lipschitz–Picard type theorem (see, e.g., [47]), the flows $\xi_{s,t}$ and $\xi_{t,s}$ exist, are unique and are $\mathscr{C}^1$-time curves in the group of $\mathscr{C}^{k,\gamma}$-diffeomorphisms of M. Let $\beta_{\mathrm{in}} = d\alpha_{\mathrm{in}} \in \mathscr{C}^{1,\gamma}\Lambda^p(M)$ be any given exact p-form, with $\alpha_{\mathrm{in}} \in \mathscr{C}^{2,\gamma}\Lambda^{p-1}(M)$. Let $\beta \in L^\infty([0,T];\mathscr{C}^{1,\gamma}\Lambda^p(M)) \cap \mathrm{Lip}([0,T];\mathscr{C}^{0,\gamma}\Lambda^p(M))$ be a time-dependent family of exact p-forms, i.e., $\beta = d\alpha$ with $\alpha \in L^\infty([0,T];\mathscr{C}^{2,\gamma}\Lambda^{p-1}(M)) \cap \mathrm{Lip}([0,T];\mathscr{C}^{1,\gamma}\Lambda^{p-1}(M))$.

Then, β satisfies the following Lie transport equation,

$$\partial_t\beta_t(x) + \pounds_{V_0(t,x)}\beta_t(x) = 0, \quad x \in M, \quad t \in]0,T], \tag{13}$$

with $\beta_0(x) = \beta_{\mathrm{in}}(x)$ as initial data, if and only if, for all $x \in M$, and for all s such that $0 \le s \le t \le T$, the following backward generalized Cauchy invariants equation,

$$\beta_t(x) = d(\alpha_{i_1\ldots i_{p-1}}(s) \circ \xi_{t,s}(x)) \wedge d\xi_{t,s}^{i_1}(x) \wedge \cdots \wedge d\xi_{t,s}^{i_{p-1}}(x), \tag{14}$$

is satisfied.

From Theorem 2 we can infer the following corollary.

Corollary 2 (*The Kelvin Circulation Theorem and the Helmholtz Flux Conservation Theorem*)**.** *Under the same hypotheses as in Theorem 2, for any p-chain c of M, we obtain*

$$\oint_{\partial c}\alpha_t = \oint_{\xi_{t,s}(\partial c)}\alpha_s, \qquad \forall s \ \ s.t. \ \ 0 \le s \le t \le T, \tag{15}$$

$$\int_c \beta_t = \int_{\xi_{t,s}(c)}\beta_s, \qquad \forall s \ \ s.t. \ \ 0 \le s \le t \le T. \tag{16}$$

2.2. Proofs of the results of Section 2.1

In this section, we give the proof of the results of the previous section, namely Theorem 2, Corollary 2, Theorem 1 and Corollary 1.

Proof of Theorem 2. The proof is similar to the proof of Theorem 1 of [20], but for the sake of completeness we supply it. For $0 \le s \le t \le T$, we set $x = \xi_{s,t}(y)$ and its inverse $y = \xi_{t,s}(x)$, where of course $x, y \in M$. Since β is Lie-advected, by the Lie derivative theorem (see, e.g., Theorem 5.4.5 in [3]), we have $\beta_s(y) = (\xi_{s,t})^*\beta_t(x)$ or $\beta_t(x) = (\xi_{s,t})_*\beta_s(y)$, where $(\xi_{s,t})^*$ (resp. $(\xi_{s,t})_*$) denotes the pullback (resp. the pushforward) operator associated with the flow $\xi_{s,t}$. For precise definitions, especially in local coordinates, of the pullback and pushforward operators associated with a flow map, we refer the reader to [3,36]. We start by writing $(\xi_{s,t})_*\beta_s$ in terms of its components in the x-coordinate. Using the commutation property between the exterior derivative and the pushforward operator (i.e., $\xi_* d\alpha = d\xi_*\alpha$) and the homomorphism $(\xi_{t,s})_*$ on the graded algebra $\Lambda^p(M)$ (i.e., $\xi_*(\alpha \wedge \beta) = \xi_*\alpha \wedge \xi_*\beta$), we obtain

$$\begin{aligned} \beta_t(x) &= (\xi_{s,t})_*\beta_s(y) \\ &= (\xi_{s,t})_* \sum_{j_1<\cdots<j_p} \beta_{j_1\ldots j_p}(s,y)\, dy^{j_1} \wedge \cdots \wedge dy^{j_p} \\ &= \frac{1}{p!}\big((\xi_{s,t})_*\beta_{j_1\ldots j_p}(s,y)\big)\, d\big((\xi_{s,t})_* y^{j_1}\big) \wedge \cdots \wedge d\big((\xi_{s,t})_* y^{j_p}\big) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{p!}\beta_{j_1\dots j_p}(s,\xi_{t,s}(x))\, d\xi^{j_1}_{t,s}(x)\wedge\cdots\wedge d\xi^{j_p}_{t,s}(x)\\
&= \frac{1}{p!}\frac{\partial y^{j_1}}{\partial x^{i_1}}\cdots\frac{\partial y^{j_p}}{\partial x^{i_p}}\,\beta_{j_1\dots j_p}(s,\xi_{t,s}(x))\, dx^{i_1}\wedge\cdots\wedge dx^{i_p}\\
&= \sum_{i_1<\cdots<i_p}((\xi_{s,t})_*\beta_s)_{i_1\dots i_p}(x)\, dx^{i_1}\wedge\cdots\wedge dx^{i_p}.
\end{aligned}
\tag{17}
$$

Next, using the generalized Kronecker symbol $\delta^{i_1\dots i_p}_{j_1\dots j_p}$ (also noted $\varepsilon^{i_1\dots i_p}_{j_1\dots j_p}$ in the literature, see, e.g., [36]) defined by

$$
\delta^{i_1\dots i_p}_{j_1\dots j_p} = \begin{cases} 0 & \text{if } (i_1\dots i_p) \text{ is not a permutation of } (j_1\dots j_p)\\ +1 & \text{if } (i_1\dots i_p) \text{ is an even permutation of } (j_1\dots j_p)\\ -1 & \text{if } (i_1\dots i_p) \text{ is an odd permutation of } (j_1\dots j_p),\end{cases}
$$

and the relation $\beta = d\alpha$, we obtain

$$
\begin{aligned}
\beta(x) &= d\alpha(x)\\
&= d\sum_{i_1<\cdots<i_{p-1}}\alpha_{i_1\dots i_{p-1}}(x)\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_{p-1}}\\
&= \sum_{i_1<\cdots<i_{p-1}} d\alpha_{i_1\dots i_{p-1}}(x)\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_{p-1}}\\
&= \sum_{i_1<\cdots<i_{p-1}} \frac{\partial}{\partial x^k}\alpha_{i_1\dots i_{p-1}}(x) dx^k\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_{p-1}}\\
&= \sum_{i_1<\cdots<i_{p-1}} \delta^{k i_1\dots i_{p-1}}_{j_1\dots j_p}\frac{\partial}{\partial x^k}\alpha_{i_1\dots i_{p-1}}(x) dx^{j_1}\wedge\cdots\wedge dx^{j_p},
\end{aligned}
$$

from which we deduce

$$
\beta_{l_1\dots l_p}(x) = \delta^{k i_1\dots i_{p-1}}_{l_1\dots l_p}\frac{\partial}{\partial x^k}\alpha_{i_1\dots i_{p-1}}(x). \tag{18}
$$

Substituting (18) into (17) we obtain

$$
\begin{aligned}
\beta_t(x) &= \frac{1}{p!}\delta^{j_p j_1\dots j_{p-1}}_{l_1\dots l_p}\frac{\partial y^{l_1}}{\partial x^{i_1}}\cdots\frac{\partial y^{l_p}}{\partial x^{i_p}}\frac{\partial}{\partial y^{j_p}}\alpha_{j_1\dots j_{p-1}}(s,\xi_{t,s}(x))\, dx^{i_1}\wedge\cdots\wedge dx^{i_p}\\
&= \frac{1}{p!}\delta^{j_p j_1\dots j_{p-1}}_{l_1\dots l_p}\frac{\partial y^{l_1}}{\partial x^{i_1}}\cdots\frac{\partial y^{l_p}}{\partial x^{i_p}}\frac{\partial x^k}{\partial y^{j_p}}\frac{\partial}{\partial x^k}\big(\alpha_{j_1\dots j_{p-1}}(s,\xi_{t,s}(x))\big)\, dx^{i_1}\wedge\cdots\wedge dx^{i_p}\\
&= \frac{1}{p!}(-1)^{p-1}\delta^{j_1\dots j_p}_{l_1\dots l_p}\frac{\partial y^{l_1}}{\partial x^{i_1}}\cdots\frac{\partial y^{l_p}}{\partial x^{i_p}}\frac{\partial x^k}{\partial y^{j_p}}\frac{\partial}{\partial x^k}\big(\alpha_{j_1\dots j_{p-1}}(s,\xi_{t,s}(x))\big)\, dx^{i_1}\wedge\cdots\wedge dx^{i_p}.
\end{aligned}
\tag{19}
$$

Using now the Laplace expansion of determinants, we may define recursively

$$
\begin{aligned}
\delta^{j_1\dots j_p}_{i_1\dots i_p} &= \begin{vmatrix} \delta^{j_1}_{i_1} & \dots & \delta^{j_1}_{i_p}\\ \vdots & \ddots & \vdots\\ \delta^{j_p}_{i_1} & \dots & \delta^{j_p}_{i_p}\end{vmatrix}\\
&= \sum_{k=1}^{p}(-1)^{p+k}\delta^{j_p}_{i_k}\delta^{j_1\dots j_k\dots\widehat{j_p}}_{i_1\dots\widehat{i_k}\dots i_p},
\end{aligned}
\tag{20}
$$

where the hat indicates an omitted index in the sequence. Using (20), Eq. (19) becomes

$$
\begin{aligned}
\beta_t(x) &= \frac{1}{p!}\sum_{n=1}^{p}(-1)^{n-1}\delta^{j_p}_{l_n}\delta^{j_1\dots j_{p-1}}_{l_1\dots\widehat{l_n}\dots l_p}\frac{\partial y^{l_1}}{\partial x^{i_1}}\cdots\frac{\partial y^{l_p}}{\partial x^{i_p}}\frac{\partial x^k}{\partial y^{j_p}}\frac{\partial}{\partial x^k}\big(\alpha_{j_1\dots j_{p-1}}(s,\xi_{t,s}(x))\big)\, dx^{i_1}\wedge\cdots\wedge dx^{i_p}\\
&= \frac{1}{p!}\sum_{n=1}^{p}(-1)^{n-1}\delta^{j_1\dots j_{p-1}}_{l_1\dots\widehat{l_n}\dots l_p}\frac{\partial y^{l_1}}{\partial x^{i_1}}\cdots\frac{\partial y^{l_p}}{\partial x^{i_p}}\frac{\partial x^k}{\partial y^{l_n}}\frac{\partial}{\partial x^k}\big(\alpha_{j_1\dots j_{p-1}}(s,\xi_{t,s}(x))\big)\, dx^{i_1}\wedge\cdots\wedge dx^{i_p}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{p!}\sum_{n=1}^{p}(-1)^{n-1}\delta^{j_1\cdots j_{p-1}}_{l_1\ldots\widehat{l_n}\ldots l_p}\frac{\partial y^{l_1}}{\partial x^{i_1}}\cdots\widehat{\frac{\partial y^{l_n}}{\partial x^{i_n}}}\cdots\frac{\partial y^{l_p}}{\partial x^{i_p}}\delta^k_{i_n}\frac{\partial}{\partial x^k}\big(\alpha_{j_1\ldots j_{p-1}}(s,\xi_{t,s}(x))\big)\,dx^{i_1}\wedge\cdots\wedge dx^{i_p}\\
&= \frac{1}{p!}\sum_{n=1}^{p}(-1)^{n-1}\delta^{j_1\cdots j_{p-1}}_{l_1\ldots\widehat{l_n}\ldots l_p}\frac{\partial y^{l_1}}{\partial x^{i_1}}\cdots\widehat{\frac{\partial y^{l_n}}{\partial x^{i_n}}}\cdots\frac{\partial y^{l_p}}{\partial x^{i_p}}\frac{\partial}{\partial x^{i_n}}\big(\alpha_{j_1\ldots j_{p-1}}(s,\xi_{t,s}(x))\big)\,dx^{i_1}\wedge\cdots\wedge dx^{i_p}\\
&= \frac{1}{p!}\sum_{n=1}^{p}(-1)^{n-1}\delta^{j_1\cdots j_{p-1}}_{l_1\ldots\widehat{l_n}\ldots l_p}\Big(\frac{\partial y^{l_1}}{\partial x^{i_1}}dx^{i_1}\Big)\wedge\cdots\wedge\Big(\frac{\partial}{\partial x^{i_n}}\big(\alpha_{j_1\ldots j_{p-1}}(s,\xi_{t,s}(x))\big)dx^{i_n}\Big)\wedge\\
&\quad\cdots\wedge\Big(\frac{\partial y^{l_p}}{\partial x^{i_p}}dx^{i_p}\Big)\\
&= \frac{1}{p!}\sum_{n=1}^{p}\delta^{j_1\cdots j_{p-1}}_{l_1\ldots\widehat{l_n}\ldots l_p}\,d(\alpha_{j_1\ldots j_{p-1}}(s)\circ\xi_{t,s}(x))\wedge dy^{l_1}\wedge\cdots\wedge\widehat{dy^{l_n}}\wedge\cdots\wedge dy^{l_p}\\
&= \frac{1}{(p-1)!}\delta^{j_1\cdots j_{p-1}}_{l_1\ldots l_{p-1}}\,d(\alpha_{j_1\ldots j_{p-1}}(s)\circ\xi_{t,s}(x))\wedge dy^{l_1}\wedge\ldots\wedge dy^{l_{p-1}}\\
&= \frac{1}{(p-1)!}\delta^{j_1\cdots j_{p-1}}_{l_1\ldots l_{p-1}}\,d(\alpha_{j_1\ldots j_{p-1}}(s)\circ\xi_{t,s}(x))\wedge d\xi^{l_1}_{t,s}(x)\wedge\ldots\wedge d\xi^{l_{p-1}}_{t,s}(x)\\
&= d(\alpha_{l_1\ldots l_{p-1}}(s)\circ\xi_{t,s}(x))\wedge d\xi^{l_1}_{t,s}(x)\wedge\ldots\wedge d\xi^{l_{p-1}}_{t,s}(x).
\end{aligned}
$$

In the last line of the previous equation, we have used the following identity,

$$
\frac{1}{(p-1)!}\delta^{j_1\cdots j_{p-1}}_{l_1\ldots l_{p-1}}\alpha_{j_1\ldots j_{p-1}}=\alpha_{l_1\ldots l_{p-1}},
$$

which is justified because $\alpha_{j_1\cdots j_{p-1}}$ is skew-symmetric. This ends the proof. □

Proof of Corollary 2. We start by showing formula (16). Using the commutation property between the exterior derivative and the pullback operator (i.e., $\xi^*d\alpha = d\xi^*\alpha$) and the homomorphism $(\xi_{t,s})^*$ on the graded algebra $\Lambda^p(M)$ (i.e., $\xi^*(\alpha\wedge\beta)=\xi^*\alpha\wedge\xi^*\beta$), the integration of the generalized Cauchy invariants Eq. (14) on a p-chain c gives

$$
\begin{aligned}
\int_c\beta_t(x) &= \int_c d\big(\alpha_{i_1\ldots i_{p-1}}(s)\circ\xi_{t,s}(x)\big)\wedge d\xi^{i_1}_{t,s}(x)\wedge\ldots\wedge d\xi^{i_{p-1}}_{t,s}(x)\\
&= \int_c d\big((\xi_{t,s})^*\alpha_{i_1\ldots i_{p-1}}(s,y)\big)\wedge d\big((\xi_{t,s})^*y^{i_1}\big)\wedge\ldots\wedge d\big((\xi_{t,s})^*y^{i_{p-1}}\big)\\
&= \int_c(\xi_{t,s})^*\big(d\alpha_{i_1\ldots i_{p-1}}(s,y)\wedge dy^{i_1}\wedge\ldots\wedge dy^{i_{p-1}}\big)\\
&= \int_c(\xi_{t,s})^*d\alpha_s(y) = \int_c(\xi_{t,s})^*\beta_s(y) = \int_{\xi_{t,s}(c)}\beta_s(y),
\end{aligned}
$$

where in the last equality we have used the theorem of change of variables (see, e.g., Theorem 7.1.7 in [3]). This proves formula (16). We continue with the proof of formula (15). Using the Stokes theorem on chains (see, e.g., Theorem 7.2.25 in [3]), we obtain

$$
\begin{aligned}
\oint_{\partial c}\alpha_t(x) &= \int_c d\alpha_t(x) = \int_c\beta_t(x)\\
&= \int_c d\big(\alpha_{i_1\ldots i_{p-1}}(s)\circ\xi_{t,s}(x)\big)\wedge d\xi^{i_1}_{t,s}(x)\wedge\ldots\wedge d\xi^{i_{p-1}}_{t,s}(x)\\
&= \int_c d\big(\alpha_{i_1\ldots i_{p-1}}(s)\circ\xi_{t,s}(x)\,d\xi^{i_1}_{t,s}(x)\wedge\ldots\wedge d\xi^{i_{p-1}}_{t,s}(x)\big)\\
&= \oint_{\partial c}\alpha_{i_1\ldots i_{p-1}}(s)\circ\xi_{t,s}(x)\,d\xi^{i_1}_{t,s}(x)\wedge\ldots\wedge d\xi^{i_{p-1}}_{t,s}(x)
\end{aligned}
$$

$$= \oint_{\partial c} (\xi_{t,s})^* \alpha_s(y) = \oint_{\xi_{t,s}(\partial c)} \alpha_s(y),$$

which ends the proof of (15). $\square$

Proof of Theorem 1. We begin by showing the sufficient condition. The idea of the proof is to evaluate the expectation of the difference $\beta_t - (\xi_{t,s})^*\beta_s$ by using the Itô's second formula for the pullback operator $(\xi_{t,s})^*$ acting on p-forms (see Section III.4 of [108] or Section 4.9 of [109]), where the associated flow $\xi_{t,s}$ is the unique solution of the backward Stratonovich's stochastic differential Eq. (8). From hypotheses of Theorem 1, and using Theorems III.4.1 and III.4.3 of [108], or Theorem 4.9.3 and Corollary 4.9.4 of [109] (see also [106,114]), the Itô's second formula gives,

$$\begin{aligned}
\beta_t - (\xi_{t,s})^*\beta_s &= -\big((\xi_{t,s})^*\beta_s - (\xi_{t,t})^*\beta_t\big) \\
&= -\Big(-\int_s^t \mathrm{d}r\, (\xi_{t,r})^* \partial_r \beta_r - \int_s^t \mathrm{d}r\, (\xi_{t,r})^* \big(\mathcal{L}_{V_0(r)}\beta_r\big) \\
&\qquad - \sum_{\ell=1}^m \int_s^t (\xi_{t,r})^* \big(\mathcal{L}_{V_\ell(r)}\beta_r\big)\, \hat{\mathrm{d}}W^\ell(r) \\
&\qquad + \int_s^t \mathrm{d}r\, (\xi_{t,r})^* \Big(\tfrac{1}{2}\sum_{\ell=1}^m \mathcal{L}_{V_\ell(r)}\mathcal{L}_{V_\ell(r)}\beta_r\Big)\Big) \\
&= \sum_{\ell=1}^m \int_s^t (\xi_{t,r})^* \big(\mathcal{L}_{V_\ell(r)}\beta_r\big)\, \hat{\mathrm{d}}W^\ell(r) \\
&\quad + \int_s^t \mathrm{d}r\, (\xi_{t,r})^* \Big(\partial_r\beta_r + \mathcal{L}_{V_0(r)}\beta_r - \tfrac{1}{2}\sum_{\ell=1}^m \mathcal{L}_{V_\ell(r)}\mathcal{L}_{V_\ell(r)}\beta_r\Big).
\end{aligned} \tag{21}$$

Since we have the following martingale property,

$$\mathbb{E}\Big[\int_s^t \sum_{\ell=1}^m (\xi_{t,r})^* \big(\mathcal{L}_{V_\ell(r)}\beta_r\big)\, \hat{\mathrm{d}}W^\ell(r)\Big] = 0, \tag{22}$$

if now we assume that, under assumptions (H1) – (H3), $\beta \in L^\infty([0,T], \mathscr{C}^{2,\mu}\Lambda^p(M)) \cap \mathrm{Lip}([0,T], \mathscr{C}^{0,\mu}\Lambda^p(M))$ satisfies the equation,

$$\partial_t\beta_t + \mathcal{L}_{V_0(t)}\beta_t - \tfrac{1}{2}\sum_{\ell=1}^m \mathcal{L}_{V_\ell(t)}\mathcal{L}_{V_\ell(t)}\beta_t = 0, \qquad \text{on} \quad M, \quad t \in]0,T], \tag{23}$$

with initial condition $\beta_0 = \beta_{\mathrm{in}} \in \mathscr{C}^{2,\mu}\Lambda^p(M)$, then we obtain from the expectation of (21), $\beta_t = \mathbb{E}[(\xi_{t,s})^*\beta_s]$. On the one hand, we have $\mathbb{E}[(\xi_{t,s})^*\beta_s] = \mathbb{E}[(\xi_{s,t})_*\beta_s]$, and on the other hand, following the proof of Theorem 2, we have the algebraic identity, $(\xi_{s,t})_*\beta_s = d(\alpha_{i_1,\ldots,i_{p-1}}(s) \circ \xi_{t,s}) \wedge d\xi_{t,s}^{i_1} \wedge \cdots \wedge d\xi_{t,s}^{i_{p-1}}$, since $\beta_s = d\alpha_s$ (for all $s \in [0,T]$) and because, from Proposition 1, the stochastic flow $\xi_{s,t}$ is sufficiently smooth in the space variable for all times $0 \le s \le t \le T$ (at least $\xi_{s,t} \in \mathscr{C}([0,T]\times[0,T]; \mathscr{C}^{1,\mu}(M))$, with $0 < \mu < \gamma < 1$). Therefore, we obtain

$$\beta_t = \mathbb{E}\big[d(\alpha_{i_1,\ldots,i_{p-1}}(s) \circ \xi_{t,s}) \wedge d\xi_{t,s}^{i_1} \wedge \cdots \wedge d\xi_{t,s}^{i_{p-1}}\big]. \tag{24}$$

We now show the necessary condition. Reciprocally, we assume (24), that is, $\beta_t = \mathbb{E}[(\xi_{t,s})^*\beta_s]$, for all $0 \le s \le t \le T$, where $\beta_0 = \beta_{\mathrm{in}} = d\alpha_{\mathrm{in}} \in \mathscr{C}^{2,\mu}\Lambda^p(M)$ (or $\alpha_{\mathrm{in}} \in \mathscr{C}^{3,\mu}\Lambda^p(M)$) and where the stochastic flow of diffeomorphisms $\xi_{t,s} \in \mathscr{C}([0,T] \times [0,T]; \mathscr{C}^{k,\mu}(M))$, with $k \ge 3$ and $0 < \mu < \gamma < 1$, is the unique regular solution of the backward Stratonovich's stochastic differential Eq. (8), with hypotheses (H1) – (H3) as in Theorem 1. In particular, we have

$$\beta_t = \mathbb{E}\big[d(\alpha_{0\, i_1,\ldots,i_{p-1}} \circ \xi_{t,0}) \wedge d\xi_{t,0}^{i_1} \wedge \cdots \wedge d\xi_{t,0}^{i_{p-1}}\big] = \mathbb{E}[(\xi_{t,0})^*\beta_0]. \tag{25}$$

Since $\beta_0 = d\alpha_0 \in \mathscr{C}^{2,\mu}\Lambda^p(M)$ and $d\xi^i_{t,0} \in \mathscr{C}([0,T];\mathscr{C}^{k-1,\mu}(M))$, with $k \geq 3$, we then obtain from (25), $\beta \in \mathscr{C}([0,T],\mathscr{C}^{2,\mu}\Lambda^p(M))$. Moreover, we have $\beta \in \mathrm{Lip}([0,T],\mathscr{C}^{0,\mu}\Lambda^p(M))$. Indeed, for any $\eta \in \mathscr{C}^{2,\mu}\Lambda^p(M)$, the Itô's first formula (see, e.g., Theorem III.4.1 of [108]) gives

$$(\xi_{t,s})^*\eta - \eta = \\ -\sum_{\ell=1}^m \int_s^t \pounds_{V_\ell(r)}\big((\xi_{r,s})^*\eta\big)\,\mathrm{d}W^\ell(r) + \int_s^t \mathrm{d}r\,\Big(\tfrac{1}{2}\sum_{\ell=1}^m \pounds_{V_\ell(r)}\pounds_{V_\ell(r)} - \pounds_{V_0(r)}\Big)(\xi_{r,s})^*\eta. \tag{26}$$

Using (25), and the martingale property,

$$\mathbb{E}\Big[\sum_{\ell=1}^m \int_0^t \pounds_{V_\ell(r)}\big((\xi_{r,0})^*\beta_0\big)\,\mathrm{d}W^\ell(r)\Big] = 0,$$

the expectation of (26), where we take $s = 0$ and $\eta = \beta_0$, gives

$$\beta_t - \beta_0 = \int_0^t \mathrm{d}r\,\Big(\tfrac{1}{2}\sum_{\ell=1}^m \pounds_{V_\ell(r)}\pounds_{V_\ell(r)} - \pounds_{V_0(r)}\Big)\beta_r. \tag{27}$$

From assumption (H2) and $\beta \in \mathscr{C}([0,T],\mathscr{C}^{2,\mu}\Lambda^p(M))$, the integrand of the time integral in the right-hand side of (27) is integrable in time since it belongs to $L^\infty([0,T],\mathscr{C}^{0,\mu}\Lambda^p(M))$. Therefore, Eq. (27) is well defined and proves immediately that $\beta \in \mathrm{Lip}([0,T],\mathscr{C}^{0,\mu}\Lambda^p(M))$ with $\lim_{t\to 0}\beta_t = \beta_0 = \beta_{\mathrm{in}}$ in $\mathscr{C}([0,T]\times M)$. We are now ready to prove (23) from (24). From (24), i.e., $\beta_t = \mathbb{E}[(\xi_{t,s})^*\beta_s]$, by using the Itô's second formula (21) and the martingale property (22), we obtain

$$0 = \frac{\beta_t - \mathbb{E}[(\xi_{t,s})^*\beta_s]}{t-s} = \mathbb{E}\Big[\frac{1}{t-s}\int_s^t \mathrm{d}r\,(\xi_{t,r})^*\Big(\partial_r\beta_r + \pounds_{V_0(r)}\beta_r - \tfrac{1}{2}\sum_{\ell=1}^m \pounds_{V_\ell(r)}\pounds_{V_\ell(r)}\beta_r\Big)\Big]. \tag{28}$$

From $\beta \in \mathscr{C}([0,T],\mathscr{C}^2\Lambda^p(M)) \cap \mathrm{Lip}([0,T],\mathscr{C}^{0,\mu}\Lambda^p(M))$ and assumption (H2), the integrand of the time integral in (28) is Lebesgue-integrable in time on the time interval $[0,T]$, with values in $\mathscr{C}^{0,\mu}\Lambda^p(M)$. Therefore, using the Lebesgue differentiation theorem (see, e.g., Corollary 1 in §1.3 of Chapter I of [156]) we can pass to the limit $s \to t$ in (28), and we obtain Eq. (23) with the initial condition $\beta_0 = \beta_{\mathrm{in}}$. We can also obtain (23) from (27). Indeed, substracting (27) to itself after substituting s to t, we can pass to the limit $s \to t$ in the result by using $\beta \in \mathscr{C}([0,T],\mathscr{C}^2\Lambda^p(M)) \cap \mathrm{Lip}([0,T],\mathscr{C}^{0,\mu}\Lambda^p(M))$ and assumption (H2). The limit gives (23), which ends the proof of Theorem 1. □

Proof of Corollary 1. Substituting to the generalized Cauchy invariants Eq. (14) the statistical generalized Cauchy invariants Eq. (10), the proof Corollary 1 is the same as the proof of Corollary 2. □

3. First application: Lagrangian formulation of non-ideal hydrodynamic and magnetohydrodynamic models

Here, we show how our general result, namely Theorem 1, can be used to obtain novel (stochastic) Lagrangian formulations for hydrodynamic and magnetohydrodynamic dissipative flows. For this, we first need to link the sum of squares of Lie derivative operators to the Hodge–De Rham Laplacian operator, by embedding a closed manifold M isometrically as a submanifold of some Euclidean space. This first task is achieved in Section 3.1, where the concept of gradient Brownian systems is introduced [49–54]. In Section 3.2, we recover and generalize the result of Constantin–Iyer [37,38] for the 3-dimensional incompressible Navier–Stokes equations on the Euclidean space. In Sections 3.3 and 3.4, we use Theorem 1 for magnetohydrodynamic flows in order to obtain new Lagrangian formulations of the non-ideal incompressible extended MHD equations and of the non-ideal two-fluid model respectively.

3.1. Gradient Brownian systems and diffusion by embeddings

Let M be a Riemannian manifold satisfying hypothesis (H1). Let $\mathfrak{e} : M \to \mathbb{R}^m$ be a $\mathscr{C}^{\mathfrak{p}}$-isometric embedding of M into the Euclidean space $\mathbb{R}^m$ with $m \geq n$, given by $\mathfrak{e} = (\mathfrak{e}_1(x), \ldots, \mathfrak{e}_m(x))$. For the definition of (isometric) embeddings, the reader can consult, e.g., [3,22,43]. The existence of such an isometric embedding is ensured by the Nash isometric embedding theorem, namely Theorem 2 of [135] (see also Theorem 4.34 of [12]). For more details on the embedding problem in differential geometry, we refer the reader to some original papers [134,135,163,164], and also to some textbooks [3,12,22,43]. Let $\mathcal{P}(x) : \mathbb{R}^m \to T_xM$ be the orthogonal projection of $\mathbb{R}^m$ onto the tangent space of M at the point x, considered as a subset of $\mathbb{R}^m$ by using the tangent map $T\mathfrak{e}(x)$ as an identification. Indeed, the injective linear map $d\mathfrak{e}(x) : T_xM \to \mathbb{R}^m$ is defined by using the tangent map $T\mathfrak{e} \in \mathcal{L}(T_xM, \mathbb{R}^m)$, which is given, in the local coordinate x, by a matrix with entries $(T\mathfrak{e}(x))_{ij} = \partial\mathfrak{e}_i(x)/\partial x^j$, for $i = 1, \ldots, m$, and $j = 1, \ldots, n$. Note that the map $d\mathfrak{e}(x)$ is also the pushforward of vectors of T_xM onto the tangent space $T_{\mathfrak{e}(x)}\mathbb{R}^m = \mathbb{R}^m$. Using the notation $d\mathfrak{e}^*(x) = (d\mathfrak{e}(x))^*$ for the adjoint map of $d\mathfrak{e}(x)$, the adjoint map $(d\mathfrak{e}(x))^* : \mathbb{R}^m \to T_x^*M$ is defined by duality through the following standard relation: $(d\mathfrak{e}^*(x)e)(X) = (d\mathfrak{e}(x)X)(e)$, for all $X \in T_xM$, and for all $e \in \mathbb{R}^m$. Let $\{e_1, \ldots, e_m\}$ be an orthonormal basis of $\mathbb{R}^m$, we then define $p_\ell(x) \in T_x^*M$ by $p_\ell(x) := d\mathfrak{e}^*(x)e_\ell$, and $\mathcal{P}_\ell(x) \in T_xM$, by $\mathcal{P}_\ell(x) := (p_\ell(x))^\sharp$ (see Notation 1). Then, for all $X \in T_xM$, we obtain by duality,

$$\begin{aligned} p_\ell(x)(X) &= (d\mathfrak{e}^*(x)e_\ell)(X) = (d\mathfrak{e}(x)X)(e_\ell) = d\mathfrak{e}_\ell(x)(X) \\ &= \langle \operatorname{grad}\mathfrak{e}_\ell(x), X\rangle_{T_xM} = \langle (p_\ell(x))^\sharp, X\rangle_{T_xM}. \end{aligned}$$

Therefore, we obtain

$$\mathcal{P}_\ell(x) = \operatorname{grad}\mathfrak{e}_\ell(x) = g^{ij}(x)\frac{\partial\mathfrak{e}_\ell}{\partial x^j}(x)\frac{\partial}{\partial x^i}, \quad x \in M, \quad \ell = 1, \ldots, m. \tag{29}$$

This is why the system $\{\mathcal{P}_\ell, W^\ell\}_{\ell=1}^m$ is often called a gradient Brownian system [49].

We next present a simple and fundamental example, which is the isometric embedding of the n-dimensional sphere $\mathbb{S}^n$ into the Euclidean space $\mathbb{R}^{n+1}$ (see, e.g., [60,91]).

Example 1 (*Embedding of the n-Dimensional Sphere $\mathbb{S}^n$ Into $\mathbb{R}^{n+1}$*)**.** If $\mathbb{S}^n$ is the sphere of dimension n, then its tangent space $T\mathbb{S}^n_x$ can be identified with

$$T\mathbb{S}^n_x = \left\{ v \in \mathbb{R}^{n+1} \mid \langle v, x\rangle_{\mathbb{R}^{n+1}} = 0 \right\}.$$

The standard embedding of the sphere $\mathbb{S}^n$ into $\mathbb{R}^{n+1}$ is characterized by the following orthogonal projection $\mathcal{P}(x) : \mathbb{R}^{n+1} \to T\mathbb{S}^n_x$, defined by

$$\mathcal{P}(x)w = w - \langle w, x\rangle_{\mathbb{R}^{n+1}}\, x, \quad w \in \mathbb{R}^{n+1}.$$

Let $\{e_1, \ldots, e_{n+1}\}$ be an orthogonal basis of $\mathbb{R}^{n+1}$, then

$$\mathcal{P}_\ell(x) = e_\ell - \langle e_\ell, x\rangle_{\mathbb{R}^{n+1}}\, x = e_\ell - x_\ell x.$$

In other words, for each $x \in M$, $\mathcal{P}(x) \in \mathcal{L}(\mathbb{R}^{n+1}; T\mathbb{S}^n_x)$ is a linear map from $\mathbb{R}^{n+1}$ to $T\mathbb{S}^n_x$, which can be represented by a square matrix with the following entries,

$$\mathcal{P}_{ij}(x) = \delta_{ij} - x_i x_j, \quad i, j \in \{1, \ldots, n+1\}, \quad \forall\, x \in \mathbb{S}^n \subset \mathbb{R}^{n+1}.$$

We end this section by recalling a lemma on the Hodge–De Rham Laplacian operator for p-forms on manifolds, which is important for the applications that we develop below.

Lemma 1 (*The Hodge–De Rham Laplacian in Terms of Isometric Embeddings*)**.** *Let M be a Riemannian manifold satisfying hypothesis* (H1)*. Let $\mathfrak{e} : M \to \mathbb{R}^m$ be a $\mathscr{C}^{\mathfrak{p}}$-isometric embedding of M into $\mathbb{R}^m$ with $m \geq n$, whose existence is ensured by the Nash isometric embedding theorem (Theorem 2 of [135]). Let*

$$\begin{cases} \mathcal{P}(x) : \mathbb{R}^m & \longrightarrow \quad T_x M \\ \qquad\qquad X & \longrightarrow \quad \mathcal{P}(x)X, \end{cases}$$

be the orthogonal projection of $\mathbb{R}^m$ into $T_x M$, for all $x \in M$, given by $\mathcal{P}_\ell = \operatorname{grad} \mathfrak{e}_\ell$, with $\ell \in \{1, \ldots, m\}$. Then, for any $\beta \in \mathscr{C}^2 \Lambda^p(M)$,

$$\Delta_{\mathrm{H}} \beta = \sum_{\ell=1}^{m} \pounds_{\mathcal{P}_\ell} \pounds_{\mathcal{P}_\ell} \beta, \tag{30}$$

with $\Delta_{\mathrm{H}} := -(dd^* + d^* d)$, the Hodge–De Rham Laplacian.

Proof. The proof can be found in Addendum C4 of [50], or in Appendix of [54] or else in Section 2.3 of [52] (see also [10,60] and Section 4.2 of [157]). ☐

Remark 7 (*What is the Right Laplacian on Riemannian Manifolds?*)**.** Note that for diffusive equations on manifolds, which involve a Laplacian, the question of which is the right Laplacian to use is not always obvious, especially since this choice can strongly impact the properties and the mathematical analysis of such equations. We refer the reader to [10,35,47,60,128,139,159] for such a problem in the case of the Navier–Stokes equations. There are at least two standard Laplacian operators for p-forms on manifolds. The first one is the Hodge–De Rham operator defined in Lemma 1. The second one, called the Laplace–Beltrami operator (when it acts on scalar function) or the Bochner Laplacian (when it acts on vector fields), is defined by $\Delta_{\mathrm{B}} := -\nabla^* \nabla = g^{ij} \nabla_i \nabla_j$, where ∇^* is the adjoint of ∇ (see, e.g., [159]). These two Laplacians are related by the Weitzenböck formula, which has the following generic form $\Delta_{\mathrm{H}} = \Delta_{\mathrm{B}} - \mathrm{Ric}$, where the term Ric contains Ricci curvature terms (see, e.g., [50,52,54,159]). Ebin and Marsden suggest at the end of their paper [47] that for the Navier–Stokes equations on Riemannian manifolds the natural Laplacian should be $-2\mathrm{Def}^*\mathrm{Def} = \Delta_{\mathrm{H}} + 2\mathrm{Ric} - dd^*$, where Def is the deformation operator and Def^* its adjoint (for more details see [128,159]). From our general framework of Section 2, which includes the Navier–Stokes equations as a particular case (see Section 3.2 below), we observe that the natural Laplacian for the Navier–Stokes equations is the Hodge–De Rham Laplacian. Of course, in $\mathbb{R}^n$ all these Laplacians coincide with the standard Laplacian in $\mathbb{R}^n$.

Remark 8 (*Extrinsic Versus Intrinsic Representation of the Brownian Motion*)**.** The study of stochastic differential equations on manifolds by embeddings is usually called an *extrinsic* method because it depends on the embedding of M into some Euclidean space $\mathbb{R}^m$. In such a representation, the Brownian motion on M is constructed by the projection on the tangent space (of M) of an Euclidean Brownian motion of dimension m, which is in general larger than the dimension n of the manifold M, whereas we expect that on M, the Brownian motion should still be intrinsically a n-dimensional object. There exist two other different but equivalent representations of the Brownian motion to study stochastic differential equations on manifolds. The second one, called *intrinsic* representation, makes use of the concepts of horizontal lift and stochastic development, which are central concepts in the Eells–Elworthy–Malliavin [48,91,92,123] construction of the n-dimensional Brownian motion on Riemannian manifolds. In such a construction of the Brownian motion as a n-dimensional object, a stochastic path or curve on M can be lifted to an horizontal curve (of frames) in the frame bundle of M by solving deterministic ordinary differential equations. To this horizontal curve (called stochastic development) corresponds uniquely a curve (called anti-development) in the Euclidean space of the same dimension (for instance n) where the standard n-dimensional Brownian motion is used. Therefore, there is a one-to-one correspondence between the set of curves on the manifold M of dimension

n and their anti-developments in the Euclidean space of the same dimension. The third representation is a local representation based on a well chosen coordinate system (collection of charts or atlas). Even if intrinsic representation appears more elegant and compact, for some precise mathematical analysis or numerical computations, for which local calculations are unavoidable, the extrinsic representation such as the local coordinates or the (global) embedding method seems more judicious.

3.2. The incompressible Navier–Stokes equations

3.2.1. The case of the flat torus $\mathbb{T}^3$

Here, we aim at applying Theorem 1 to the incompressible Navier–Stokes equations, written in terms of the vorticity vector $\omega(t,x) = \nabla \times u(t,x)$, where $u(t,x)$ is the divergence-free velocity vector field governed by the standard incompressible Navier–Stokes equations. The Navier–Stokes equations is posed on the 3-dimensional flat torus $\mathbb{T}^3$, which is equipped with the canonical orthonormal basis $\{e_i\}_{i=1,2,3}$. In Theorem 1, we take $m = 3$, $V_0(t,x) = u(t,x) \in \mathbb{R}^3$, and $V_\ell(t,x) = \sqrt{2\nu}e_\ell$, with $\ell \in \{1, 2, 3\}$, and ν the viscosity parameter. Using Notation 1, we consider the vorticity 2-form $\boldsymbol{\omega} := du^\flat$, which is linked to the vorticity vector ω by the relation $\boldsymbol{\omega} = *(\omega^\flat)$ or $\omega = (*\boldsymbol{\omega})^\sharp$ (see, e.g., [20]) and we take in Eq. (9) of Theorem 1 the 2-form $\beta = \boldsymbol{\omega}$. Since ν is a constant parameter, we obtain $\frac{1}{2}\sum_{\ell=1,2,3} \pounds^2_{\sqrt{2\nu}e_\ell}\boldsymbol{\omega} = \nu\Delta\boldsymbol{\omega}$ in (9), which is the standard Laplacian appearing in the Navier–Stokes equations, while the Lie-advected term is $\pounds_u\boldsymbol{\omega}$. Therefore, we obtain the Navier–Stokes equations in terms of the vorticity 2-form $\boldsymbol{\omega}$, namely, $\partial_t\boldsymbol{\omega} + \pounds_u\boldsymbol{\omega} - \nu\Delta\boldsymbol{\omega} = 0$. Since we have the commutation relation $[\sharp *, \pounds_X] = 0$ (here $\sharp \equiv (\cdot)^\sharp$, see Notation 1) where X is a divergence-free vector field (see Appendix A.6 of [20]), the vorticity vector ω satisfies the same equation as the vorticity 2-form $\boldsymbol{\omega}$, i.e.,

$$\partial_t\omega + \pounds_u\omega - \nu\Delta\omega = \partial_t\omega + \nabla(\omega \times u) - \nu\Delta\omega = \partial_t\omega + u\cdot\nabla\omega - \omega\cdot\nabla u - \nu\Delta\omega = 0. \tag{31}$$

Since the viscosity is a constant parameter, the backward Stratonovich's stochastic differential Eq. (8) reduces to the backward Itô's stochastic differential Eq. (32), where $W(t)$ denotes a 3-dimensional Brownian motion. Observing that in the backward Itô's stochastic differential Eq. (32) the stochastic term $\sqrt{2\nu}\hat{\mathrm{d}}W(s)$ does not depend on the spatial variable x, we obtain that any spatial derivative of Eq. (32) gives a deterministic equation, for instance $\nabla\dot{\xi}^i_{t,s}(x) = \nabla(u^i(s,\xi_{t,s}(x)))$, $\forall i \in \{1,2,3\}$, where the dot notation corresponds to the time derivative with respect to the second time argument, viz. s. Using this observation, the statistical generalized Cauchy invariants Eq. (10), written in terms of the vorticity vector ω instead of its 2-form $\boldsymbol{\omega}$, becomes Eq. (33), because the exterior derivative d (resp. the exterior product $\wedge$ between forms) becomes the standard Euclidean gradient ∇ (resp. the cross product $\times$ between vectors). Therefore, for the 3-dimensional flat torus $\mathbb{T}^3$, the application of Theorem 1 gives the following Lagrangian formulation for the incompressible Navier–Stokes equations on $\mathbb{T}^3$: for all s such that $0 \le s \le t \le T$, and for all $x \in \mathbb{T}^3$,

$$\hat{\mathrm{d}}\xi_{t,s}(x) = u(s,\xi_{t,s}(x))\mathrm{d}s + \sqrt{2\nu}\hat{\mathrm{d}}W(s), \tag{32}$$

$$\omega(t,x) = \sum_{i=1,2,3} \mathbb{E}\big[\nabla\dot{\xi}^i_{t,s}(x) \times \nabla\xi^i_{t,s}(x)\big], \tag{33}$$

$$\nabla \times u = \omega, \quad \nabla\cdot u = 0. \tag{34}$$

Of course, the system (32)–(34) should be completed with an initial condition that we can take at the time $t = 0$ and which can be either an initial divergence-free velocity vector field u_0, with $\nabla\cdot u_0 = 0$, or an initial vorticity ω_0, with $\nabla\cdot\omega_0 = 0$. In fact, the formulation (32)–(34) is equivalent to the Constantin–Iyer formulation [37,38] given by

$$\hat{\mathrm{d}}\xi_{t,s}(x) = u(s,\xi_{t,s}(x))ds + \sqrt{2\nu}\hat{\mathrm{d}}W(s), \quad 0 \le s \le t \le T, \quad x \in \mathbb{T}^3, \tag{35}$$

$$u(t,x) = \mathbb{E}\mathbb{P}\big[\nabla\xi_{t,s}(x)\, u(s) \circ \xi_{t,s}(x)\big], \tag{36}$$

where $\mathbb{P} \equiv \mathrm{Id} - \nabla\Delta^{-1}\nabla\cdot$ is the Leray–Hodge projection on divergence-free vector field. Indeed, after some algebra we obtain,

$$\omega(t,x) = \nabla \times u(t,x) = \nabla \times \mathbb{E}\mathbb{P}\big[\nabla\xi_{t,s}(x)\, u(s) \circ \xi_{t,s}(x)\big]$$

$$= \mathbb{E}\big[(D\xi_{t,s}(x))^{-1}\, \omega(s) \circ \xi_{t,s}(x)\big] \tag{37}$$

$$= \sum_{i=1,2,3} \mathbb{E}\big[\nabla\dot{\xi}^i_{t,s}(x) \times \nabla\xi^i_{t,s}(x)\big], \tag{38}$$

where the dot notation stands for the time derivative with respect to the second time argument, namely s. Here, we should make a remark about three different but equivalent (at least for smooth solutions) formulations of the incompressible Euler equations. Putting aside the random aspect and the expectation $\mathbb{E}$, Eq. (36) is known as the Weber's formulation of the incompressible Euler equation, while Eqs. (37) and (38) are known respectively as the Cauchy vorticity formula and the Cauchy invariants equation. In fact, the Cauchy invariants equation and the Cauchy vorticity formula, which are different but equivalent formulations of the incompressible Euler equation, are Hodge duals of each other [20].

The application of Corollary 1 gives the statistical velocity-circulation conservation theorem and the statistical vorticity-flux conservation theorem, namely, for all s such that $0 \le s \le t \le T$,

$$\oint_{\mathcal{C}} u(t,x) \cdot d\mathcal{C}(x) = \mathbb{E}\bigg[\oint_{\xi_{t,s}(\mathcal{C})} u(s,y) \cdot d\mathcal{C}(y)\bigg],$$

and

$$\int_{\mathcal{S}} \omega(t,x) \cdot d\mathcal{S}(x) = \mathbb{E}\bigg[\int_{\xi_{t,s}(\mathcal{S})} \omega(s,y) \cdot d\mathcal{S}(y)\bigg],$$

where $\mathcal{C}$ (resp. $\mathcal{S}$) is any 1-dimensional closed curve (resp. 2-dimensional surface) and $d\mathcal{C}(x)$ (resp. $d\mathcal{S}(x)$) is the local infinitesimal line (resp. surface) measure on $\mathcal{C}$ (resp. $\mathcal{S}$).

3.2.2. The case of a closed manifold M

We now apply Theorem 1 to the incompressible Navier–Stokes equations on a n-dimensional smooth closed manifold M, written is terms of the vorticity 2-form $\omega := du^\flat$ (see [20,47]), i.e.,

$$\partial_t\omega + \pounds_u\omega - \nu\Delta_{\mathrm{H}}\omega = 0, \tag{39}$$

with u the divergence-free velocity vector field such that $d^*u^\flat = 0$. For this, we take in Theorem 1, $\beta = \omega$, $V_0(t,x) = u(t,x) \in T_xM$, and $V_\ell(t,x) = \sqrt{2\nu}\mathcal{P}_\ell(x) \in T_xM$, with $\ell \in \{1,\dots,m\}$, ν the viscosity parameter, and $\mathcal{P}_\ell(x) = \operatorname{grad}\mathfrak{e}(x)$, where $\mathfrak{e}(x) : M \to \mathbb{R}^m$ is a smooth isometric embedding of the manifold M into $\mathbb{R}^m$ $(m \ge n)$. Using Lemma 1, we obtain that the term $\frac{1}{2}\sum_{1\le\ell\le m} \pounds^2_{\sqrt{2\nu}\mathcal{P}_\ell(x)}\omega$ in Eq. (9) of Theorem 1 becomes the Hodge–De Rham Laplacian $\nu\Delta_{\mathrm{H}}\omega$ appearing in the incompressible Navier–Stokes equations. Therefore, we obtain the following Lagrangian formulation for the Eulerian formulation of the incompressible Navier–Stokes Eqs. (39) on a n-dimensional smooth closed manifold M: for all s such that $0 \le s \le t \le T$, for all $x \in M$, and for all $f \in \mathscr{C}^3(M)$,

$$f(\xi_{t,s}(x)) = f(x) - \int_s^t \mathrm{d}r\, u(r) f(\xi_{t,r}(x)) - \sqrt{2\nu}\sum_{\ell=1}^m \int_s^t \mathcal{P}_\ell f(\xi_{t,r}(x)) \circ \hat{\mathrm{d}}W^\ell(r), \tag{40}$$

$$\omega(t,x) = \mathbb{E}\big[d(u^\flat_i(s) \circ \xi_{t,s}(x)) \wedge d\xi^i_{t,s}(x)\big], \tag{41}$$

$$du^\flat = \omega, \quad d^*u^\flat = 0, \tag{42}$$

where $u^\flat$ is the velocity 1-form associated with the velocity vector-field u (see Notation 1). The system (40)–(42) should be completed with an initial condition that we can take at the time $t = 0$ and which can be either an initial divergence-free velocity vector field u_0, with $d^*u_0^\flat = 0$, or an initial vorticity 2-form ω_0, with $d\omega_0 = 0$. The application of Corollary 1 still gives the statistical velocity-circulation conservation theorem and the statistical vorticity-flux conservation theorem, namely, for all s such that $0 \leq s \leq t \leq T$, and any 2-chain c of M,

$$\oint_{\partial c} u_t^\flat = \mathbb{E}\Big[\oint_{\xi_{t,s}(\partial c)} u_s^\flat\Big], \quad \text{and} \quad \int_c \omega_t = \mathbb{E}\Big[\int_{\xi_{t,s}(c)} \omega_s\Big].$$

The stochastic Lagrangian formulation (40)–(42) is written in terms of the vorticity 2-form ω. A different stochastic Lagrangian formulation, which only involves the velocity 1-form $u^\flat$ and which is closer in spirit to the original forward-in-time Constantin–Iyer formulation using Weber's formula [37], can be found in [60].

3.3. The non-ideal incompressible extended MHD equations

In order to familiarize the reader with this model, we present here the incompressible extended MHD equations on $\mathbb{R}^3$, in the non-dissipative limit, which are derived from standard two-fluid theory of plasma physics (see Section 3.4) by neglecting the displacement current, imposing quasineutrality and carrying out a systematic expansion in the mass ratio m_e/m_i, where m_e (resp. m_i) is the mass of electrons (resp. ions). The reader can consult Refs. [78,121] (see also [97]), where a detailed derivation of extended MHD from two-fluid theory is presented. Introducing the vector field

$$B_* = B + d_e^2 \nabla \times \nabla \times B,$$

the dimensionless incompressible extended MHD equations, which describe the dynamics of the divergence-free magnetic vector field $B = B(t,x)$ and velocity vector field $v = v(t,x)$, read [1,117],

$$\partial_t v + (\nabla \times v) \times v = -\nabla(p + \tfrac{1}{2}|v|^2) + (\nabla \times B) \times B_* - d_e^2 \nabla(\tfrac{1}{2}|\nabla \times B|^2), \quad \nabla \cdot v = 0, \tag{43}$$

$$\partial_t B_* + \nabla \times (B_* \times v) = -d_i \nabla \times ((\nabla \times B) \times B_*) + d_e^2 \nabla \times ((\nabla \times B) \times (\nabla \times v)), \quad \nabla \cdot B = 0, \tag{44}$$

where p is the total pressure and where d_i (resp. d_e) serves as the normalized ion (resp. electron) skin depth. In (43)–(44), setting $d_e = 0$, we recover the Hall MHD equations, while imposing $d_i = 0$, we retrieve the inertial MHD equation. Hence, extended MHD is a model that is endowed with both the Hall drift and electron inertia. Note that neither ideal nor Hall MHD are valid when one approaches length scales of the order of the electron skin depth. Hence, extended MHD constitutes a good physical model in this regime. There has been a great deal of attention in recent times focused on turbulence at small scales, i.e., scales smaller than the electron or proton skin depth. The two most notable examples in astrophysics are the Earth's magnetosphere [30,99] and the solar wind and corona [2,133,151,152]. Moreover, inclusion of the Hall and electron inertia terms has radically changed and revitalized the fields of both magnetic reconnection and turbulence, where a steepening of spectra has been shown [2,74,125,153]. The extended MHD model can also describe various instabilities, including sawtooth instability and MHD kink modes in tokamak plasmas [84], but also tearing modes induced by the presence of electron inertia, which breaks the usual frozen-in condition of ideal MHD [81]. This model is also highly relevant to study the generation of magnetic fields via the dynamo mechanism, because of the presence of an inverse cascade of magnetic helicity [28,125,126]. Further applications range from the investigation of drift-waves to low-frequency turbulence [155] and turbulent reconnection [113,154] that naturally occur in laboratory, space, astrophysical and fusion plasmas.

Using the magnetic vector potentials A, which are subjected to the Coulomb gauge, i.e., $\nabla \cdot A = 0$, we introduce the generalized velocity vector fields $v_\pm = v_\pm(t,x)$, the generalized magnetic vector potentials $A_\pm = A_\pm(t,x)$, and the generalized magnetic vector fields $B_\pm = B_\pm(t,x)$, which are defined by the following relations [41,118],

$$v_* = \nabla \times B, \quad \nabla \cdot v_* = 0, \tag{45}$$

$$
\begin{aligned}
v_{\pm} &= v - \kappa_{\mp} v_* = v - \kappa_{\mp} \nabla \times B, \quad \nabla \cdot v_{\pm} = 0, && (46)\\
A_* &= A + d_e^2 v_* = A + d_e^2 \nabla \times B, \quad \nabla \cdot A_* = 0, && (47)\\
A_{\pm} &= A_* + \kappa_{\pm} v = A + d_e^2 \nabla \times B + \kappa_{\pm} v, \quad \nabla \cdot A_{\pm} = 0, && (48)\\
B_* &= \nabla \times A_* = B + d_e^2 \nabla \times \nabla \times B = (1 - d_e^2 \Delta) B, \quad \nabla \cdot B_* = 0, && (49)\\
B_{\pm} &= \nabla \times A_{\pm} = B_* + \kappa_{\pm} \nabla \times v, \quad \nabla \cdot B_{\pm} = 0, && (50)
\end{aligned}
$$

where the parameters $\kappa_{\pm}$ are given by $\kappa_{\pm} = (d_i \pm (d_i^2 + 4d_e^2)^{1/2})/2$. If we set $d_e = 0$, then $(\kappa_+, \kappa_-) = (d_i, 0)$ and we obtain the Hall MHD equations. If we set $d_i = 0$, then $(\kappa_+, \kappa_-) = (d_e, -d_e)$ and we obtain the inertial MHD equation. Using (45)–(50), the extended MHD model (43)–(44) can be recast as Eqs. (51)–(53) below, with $\eta_{\pm} = 0$ (see, e.g., [41,118]).

3.3.1. The case of the flat torus $\mathbb{T}^3$

Using notation and definitions of the previous section, on the 3-dimensional flat torus $\mathbb{T}^3$, the non-ideal incompressible extended MHD (NI-IXMHD) equations written in a Eulerian form read

$$
\begin{aligned}
&\partial_t B_{\pm} + \nabla \times (B_{\pm} \times v_{\pm}) - \eta_{\pm} \Delta B_{\pm} = 0, && (51)\\
&v_{\pm} = \nabla \times (-\Delta)^{-1} \left(\frac{B_+ - B_-}{\kappa_+ - \kappa_-} \right) - \kappa_{\mp} \nabla \times (1 - d_e^2 \Delta)^{-1} \left(\frac{\kappa_+ B_- - \kappa_- B_+}{\kappa_+ - \kappa_-} \right), && (52)\\
&\nabla \times A_{\pm} = B_{\pm}, \quad \nabla \cdot A_{\pm} = 0, && (53)
\end{aligned}
$$

where the parameters $\eta_{\pm}$ are positive resistivity. Eqs. (51)–(52) must be completed with the initial conditions,

$$
\begin{aligned}
&v_{\pm}(0, x) = v_{\pm\,0}(x), \quad \nabla \cdot v_{\pm\,0} = 0, \quad \text{and}, && (54)\\
&B_{\pm}(0, x) = B_{\pm\,0}(x), \quad \nabla \cdot B_{\pm\,0} = 0, \quad \text{or} \quad A_{\pm}(0, x) = A_{\pm\,0}(x), \quad \nabla \cdot A_{\pm\,0} = 0. && (55)
\end{aligned}
$$

The NI-IXMHD equations, which contain dissipative cross-effects (of high-order for B, namely bi-Laplacian) in terms of original fields (v, B), are reminiscent to MHD models used in the plasmas literature [7–9,23–26,39]. Except Eqs. (52), which determine the velocity vector fields $v_{\pm}$ from the magnetic vector fields $B_{\pm}$, we observe that Eqs. (51) are structurally the same as the incompressible Navier–Stokes Eqs. (31), written in terms of the vorticity vector field. Therefore, following the same arguments as given in Section 3.2.1, where we explain how to apply Theorem 1 to the incompressible Navier–Stokes equations, the application of Theorem 1 gives the following Lagrangian formulation for the Eulerian NI-IXMHD Eqs. (51)–(53) on $\mathbb{T}^3$: for all s such that $0 \le s \le t \le T$, and for all $x \in \mathbb{T}^3$,

$$
\begin{aligned}
&\hat{\mathrm{d}} \xi_{\pm t,s}(x) = v_{\pm}(s, \xi_{\pm t,s}(x)) \mathrm{d}s + \sqrt{2\eta_{\pm}}\, \hat{\mathrm{d}} W(s), && (56)\\
&B_{\pm}(t, x) = \sum_{i=1,2,3} \mathbb{E}\big[\nabla (A_{\pm}^i(s) \circ \xi_{\pm t,s}(x)) \times \nabla \xi_{\pm t,s}^i(x) \big], && (57)\\
&v_{\pm} = \nabla \times (-\Delta)^{-1} \left(\frac{B_+ - B_-}{\kappa_+ - \kappa_-} \right) - \kappa_{\mp} \nabla \times (1 - d_e^2 \Delta)^{-1} \left(\frac{\kappa_+ B_- - \kappa_- B_+}{\kappa_+ - \kappa_-} \right), && (58)\\
&\nabla \times A_{\pm} = B_{\pm}, \quad \nabla \cdot A_{\pm} = 0, && (59)
\end{aligned}
$$

with the initial conditions (54)–(55). Applying Corollary 1, we obtain the statistical Alfvén theorem of conservation of magnetic circulation and magnetic flux: for all s such that $0 \le s \le t \le T$,

$$
\oint_{\mathcal{C}} A_{\pm}(t, x) \cdot d\mathcal{C}(x) = \mathbb{E}\left[\oint_{\xi_{t,s}(\mathcal{C})} A_{\pm}(s, y) \cdot d\mathcal{C}(y) \right],
$$

and

$$
\int_{\mathcal{S}} B_{\pm}(t, x) \cdot d\mathcal{S}(x) = \mathbb{E}\left[\int_{\xi_{t,s}(\mathcal{S})} B_{\pm}(s, y) \cdot d\mathcal{S}(y) \right],
$$

for any 1-dimensional closed curve $\mathcal{C}$, and any 2-dimensional surface $\mathcal{S}$ in $\mathbb{T}^3$.

3.3.2. The case of a closed manifold M

On a n-dimensional smooth closed manifold M, the generalization of Eqs. (51)–(53) leads to the following NI-IXMHD model,

$$\partial_t B_\pm + \pounds_{v_\pm} B_\pm - \eta_\pm \Delta_{\mathrm{H}} B_\pm = 0, \tag{60}$$

$$v_\pm^\flat = d^*(-\Delta_{\mathrm{H}})^{-1}\left(\frac{B_+ - B_-}{\kappa_+ - \kappa_-}\right) - (-1)^{n+1}\kappa_\mp d^*\big(1-(-1)^{n+1}d_e^2\Delta_{\mathrm{H}}\big)^{-1}\left(\frac{\kappa_+ B_- - \kappa_- B_+}{\kappa_+ - \kappa_-}\right), \tag{61}$$

$$dA_\pm = B_\pm, \quad d^*A_\pm = 0, \tag{62}$$

with the initial conditions,

$$v_\pm(0,x) = v_{\pm\,0}(x), \qquad d^* v^\flat_{\pm\,0} = 0, \qquad \text{and}, \tag{63}$$

$$B_\pm(0,x) = B_{\pm\,0}(x), \quad dB_{\pm\,0} = 0, \quad \text{or} \quad A_\pm(0,x) = A_{\pm\,0}(x), \quad d^*A_{\pm\,0} = 0. \tag{64}$$

Here, $v_\pm^\flat$ are the velocity 1-forms associated with the velocity vector-fields $v_\pm$ (see Notation 1), while $B_\pm$ are exact magnetic-field 2-forms deriving from the magnetic-potential 1-forms $A_\pm$. The power n in front of the Hodge–De Rham Laplacian in (61) is a resurgence of the power appearing in the definition of the exterior coderivative d^* when we rewrite the Maxwell–Ampère equation without displacement current, i.e., $v_*^\flat = *\,d*B = (-1)^{n+1}d^*B$, as we observe a few lines below (see Notation 1). In terms of the original fields, which are now the magnetic-potential 1-forms A, the magnetic-field 2-form B, and the velocity vector-field v, the fields $(v_\pm, A_\pm, B_\pm)$ are defined by $v_\pm = v - \kappa_\mp v_*$, $A_\pm = A_* + \kappa_\pm v^\flat$, and $B_\pm = dA_\pm = B_* + \kappa_\pm dv^\flat$, where $v_*^\flat = *\,d*B = (-1)^{n+1}d^*B$, $A_* = A + d_e^2 v_*^\flat$, and $B_* = dA_* = B + d_e^2\, d*d*B = (1-(-1)^{n+1}d_e^2\Delta_{\mathrm{H}})B$. Incompressibility of the fields v, v_*, $v_\pm$, A, A_*, $A_\pm$, B, B_*, and $B_\pm$ still hold, namely $dB_\pm = dB_* = dB = 0$, $d^*A_\pm = d^*A_* = d^*A = 0$, and $d^*v_\pm^\flat = d^*v_*^\flat = d^*v^\flat = 0$. Here again, we observe that Eqs. (60) are structurally the same as the incompressible Navier–Stokes Eqs. (39), written in terms of the vorticity 2-form. Therefore, following the same arguments as given in Section 3.2.2, the combination of Theorem 1 and Lemma 1 leads to the following Lagrangian formulation for the Eulerian NI-IXMHD model (60)–(62) on a n-dimensional smooth closed manifold M: for all s such that $0 \le s \le t \le T$, for all $x \in M$, and for all $f \in \mathscr{C}^3(M)$,

$$\begin{aligned}
&f(\xi_{\pm t,s}(x)) = f(x) - \int_s^t \mathrm{d}r\, v_\pm(r) f(\xi_{\pm t,r}(x)) - \sqrt{2\eta_\pm}\sum_{\ell=1}^m \int_s^t \mathcal{P}_\ell f(\xi_{\pm t,r}(x)) \circ \hat{\mathrm{d}}W^\ell(r),\\
&B_\pm(t,x) = \mathbb{E}\big[d(A_{\pm i}(s)\circ \xi_{\pm t,s}(x)) \wedge d\xi^i_{\pm t,s}(x)\big],\\
&v_\pm^\flat = d^*(-\Delta_{\mathrm{H}})^{-1}\left(\frac{B_+ - B_-}{\kappa_+ - \kappa_-}\right) - (-1)^{n+1}\kappa_\mp d^*\big(1-(-1)^{n+1}d_e^2\Delta_{\mathrm{H}}\big)^{-1}\left(\frac{\kappa_+ B_- - \kappa_- B_+}{\kappa_+ - \kappa_-}\right),\\
&dA_\pm = B_\pm, \quad d^*A_\pm = 0,
\end{aligned}$$

with the initial conditions (63)–(64). Applying Corollary 1, we obtain the following statistical Alfvén invariants,

$$\oint_{\partial c} A_{\pm t} = \mathbb{E}\left[\oint_{\xi_{\pm t,s}(\partial c)} A_{\pm s}\right], \quad \text{and} \quad \int_c B_{\pm t} = \mathbb{E}\left[\int_{\xi_{\pm t,s}(c)} B_{\pm s}\right],$$

for all s such that $0 \le s \le t \le T$, and for any 2-chain c of M.

Remark 9. It could be interesting to make a comparison between the NI-IXMHD equations of this section, written in terms of the original fields $(\omega = dv^\flat, B_*)$, with an another non-ideal IXMHD model arising from the Hamiltonian ideal IXMHD equations (namely Eqs. (51)–(53) with $\eta_\pm = 0$ and written in terms of the unknowns $(\omega = dv^\flat, B_*)$, or equivalently Eqs. (43)–(44) generalized to manifolds after taking the curl of (43)) to which we simply add standard diffusion terms such as the Laplacians $(\nu\Delta_{\mathrm{H}}\omega, \eta\Delta_{\mathrm{H}}B_*)$, i.e., without diffusion cross effects. Indeed, we then obtain $\partial_t\omega + \pounds_v\omega + \pounds_{v_*}B_* = \nu\Delta_{\mathrm{H}}\omega$ and $\partial_t B_* + \pounds_{v-d_i v_*}B_* + d_e^2\pounds_{v_*}\omega = \eta\Delta_{\mathrm{H}}B_*$.

3.4. The non-ideal two-fluid equations

3.4.1. The case of the flat torus $\mathbb{T}^3$

On the 3-dimensional flat torus $\mathbb{T}^3$, the non-ideal two-fluid model of plasma physics, written in Eulerian form [14,68,78], reads

$$m_i\Big(\partial_t(n_iu_i)+\nabla\cdot(n_iu_i\otimes u_i)\Big)=n_iq_i(E+u_i\times B)-\nabla\tilde{\pi}_i-R_{ie}+m_in_i\nu_i\Delta u_i, \tag{65}$$

$$m_e\Big(\partial_t(n_eu_e)+\nabla\cdot(n_eu_e\otimes u_e)\Big)=n_eq_e(E+u_e\times B)-\nabla\tilde{\pi}_e-R_{ei}+m_en_e\nu_e\Delta u_e, \tag{66}$$

where

$$R_{ie}=\frac{m_in_i}{\sigma_{ie}}(u_i-u_e),\qquad\text{and}\qquad R_{ei}=\frac{m_en_e}{\sigma_{ei}}(u_e-u_i). \tag{67}$$

Here, n_i, u_i and $\tilde{\pi}_i$ are respectively the density, the velocity and the scalar pressure of ions, while n_e, u_e and $\tilde{\pi}_e$ are respectively the density, the velocity and the pressure of electrons. The parameters m_i, q_i, ν_i, σ_{ie} are respectively the mass, the charge, the viscosity and the conductivity of ions. We have also the counterpart for electrons. Eqs. (65)–(67) are coupled with the Maxwell equations (without displacement current),

$$\partial_tB+\nabla\times E=0, \tag{68}$$

$$\nabla\times B=\mu_0J,\quad J=n_eq_eu_e+n_iq_iu_i, \tag{69}$$

where the magnetic vector potential A is defined by

$$\nabla\times A=B,\quad\nabla\cdot A=0. \tag{70}$$

In addition, incompressibility assumption, i.e., $\nabla\cdot u_i=\nabla\cdot u_e=0$, implies that the densities n_i and n_e are constants. Moreover, quasi-neutrality assumption $\nabla\cdot E=0$ or $Zn_i=n_e$, and $Z=1$ imply $n_i=n_e=n=$ constant and $J=ne(u_i-u_e)$. Without viscous dissipation (i.e., $\nu_i=\nu_e=0$) conservation of momentum implies $R_{ie}+R_{ei}=0$, hence $m_e/\sigma_{ei}=m_i/\sigma_{ie}$. Therefore, the system (65)–(70) becomes

$$\partial_tu_i+u_i\cdot\nabla u_i=\frac{e}{m_i}(E+u_i\times B)-\nabla\hat{\pi}_i+\nu_i\Delta u_i-\frac{u_i-u_e}{\sigma_{ie}}, \tag{71}$$

$$\partial_tu_e+u_e\cdot\nabla u_e=-\frac{e}{m_e}(E+u_e\times B)-\nabla\hat{\pi}_e+\nu_e\Delta u_e-\frac{u_e-u_i}{\sigma_{ei}}, \tag{72}$$

$$-\Delta A=\mu_0ne(u_i-u_e),\quad\nabla\cdot A=0, \tag{73}$$

where $\hat{\pi}_{i,e}=\tilde{\pi}_{i,e}/(nm_{i,e})$. Introducing the following canonical momenta,

$$p_i=u_i+\frac{e}{m_i}A,\qquad\text{and}\qquad p_e=u_e-\frac{e}{m_e}A,\qquad\text{with}\qquad\nabla\cdot p_{i,e}=0,$$

and using the parameter $\mu:=m_i/(\sigma_{ie}\mu_0ne^2)=m_e/(\sigma_{ei}\mu_0ne^2)$ and the relations $u_{i,e}\cdot\nabla u_{i,e}=\nabla(|u_{i,e}|^2/2)-u_{i,e}\times(\nabla\times u_{i,e})$, the system (71)–(73), can be recast as

$$\partial_tp_i=u_i\times(\nabla\times p_i)-\nabla\pi_i+\nu_i\Delta u_i+\mu\Delta\left(\frac{eA}{m_i}\right), \tag{74}$$

$$\partial_tp_e=u_e\times(\nabla\times p_e)-\nabla\pi_e+\nu_e\Delta u_e-\mu\Delta\left(\frac{eA}{m_e}\right), \tag{75}$$

$$u_i=p_i-\frac{e}{m_i}A,\qquad u_e=p_e+\frac{e}{m_e}A, \tag{76}$$

$$(\lambda^2-\Delta)A=\mu_0ne(p_i-p_e),\quad\nabla\cdot A=0, \tag{77}$$

where $\lambda^2:=\mu_0ne^2(1/m_e+1/m_i)$ and $\pi_{i,e}=\hat{\pi}_{i,e}+|u_{i,e}|^2/2$. To obtain (74)–(77), we have used the equation $\partial_tA+E=0$, which results from the Maxwell–Faraday Eq. (68), Eqs. (70), the quasi-neutrality assumption

$\nabla\cdot E=0$ and the periodic boundary conditions. Setting $\omega_{i,e}=\nabla\times p_{i,e}$, and choosing $\nu_i=\nu_e=\mu$, by taking the curl of Eqs. (74) and (75), the system (74)–(77) rewrites as (see [55]),

$$\partial_t\omega_{i,e}+\nabla\times(\omega_{i,e}\times u_{i,e})-\mu\Delta\omega_{i,e}=0, \tag{78}$$
$$u_i=p_i-\frac{e}{m_i}A,\qquad u_e=p_e+\frac{e}{m_e}A, \tag{79}$$
$$\nabla\times p_{i,e}=\omega_{i,e},\quad \nabla\cdot p_{i,e}=0, \tag{80}$$
$$(\lambda^2-\Delta)A=\mu_0ne(p_i-p_e),\quad \nabla\cdot A=0. \tag{81}$$

Eqs. (78)–(81) must be completed with the initial conditions $\omega_{i,e}(0,x)=\omega_{i,e\,0}(x)$, subjected to the constraints $\nabla\cdot\omega_{i,e\,0}=0$. Observing that Eqs. (78) have the same structure as the incompressible Navier–Stokes Eqs. (31), written in terms of the vorticity vector field, we can follow the same arguments as given in Section 3.2.1 and from Theorem 1, the Lagrangian formulation of the non-ideal two-fluid model (78)–(81) on $\mathbb{T}^3$ is given by the following set of equations: for all s such that $0\le s\le t\le T$, for all $x\in\mathbb{T}^3$, and $\sigma\in\{i,e\}$,

$$\hat{\mathrm{d}}\xi_{\sigma t,s}(x)=u_\sigma(s,\xi_{\sigma t,s}(x))\mathrm{d}s+\sqrt{2\mu}\,\hat{\mathrm{d}}W(s), \tag{82}$$
$$\omega_\sigma(t,x)=\sum_{i=1,2,3}\mathbb{E}\big[\nabla(p^i_\sigma(s)\circ\xi_{\sigma t,s}(x))\times\nabla\xi^i_{\sigma t,s}(x)\big], \tag{83}$$
$$\nabla\times p_\sigma=\omega_\sigma,\quad \nabla\cdot p_\sigma=0, \tag{84}$$
$$(\lambda^2-\Delta)A=\mu_0ne(p_i-p_e),\quad \nabla\cdot A=0, \tag{85}$$
$$u_i=p_i-\frac{e}{m_i}A,\qquad u_e=p_e+\frac{e}{m_e}A, \tag{86}$$

where $\xi_{\sigma t,s}(x)$ is the backward stochastic flow of particles of species σ (with $\sigma=e$ or $\sigma=i$) at time s, being at the position x at time t. Here again, Eqs. (82)–(86) have to be completed with the initial conditions $\omega_{i,e}(0,x)=\omega_{i,e\,0}(x)$, subjected to the constraints $\nabla\cdot\omega_{i,e\,0}=0$. Applying Corollary 1, we obtain the following statistical Kelvin–Helmholtz theorem of conservation of circulation and flux: for all s such that $0\le s\le t\le T$, and $\sigma\in\{i,e\}$,

$$\oint_{\mathcal{C}}p_\sigma(t,x)\cdot d\mathcal{C}(x)=\mathbb{E}\bigg[\oint_{\xi_{\sigma t,s}(\mathcal{C})}p_\sigma(s,y)\cdot d\mathcal{C}(y)\bigg],$$

and

$$\int_{\mathcal{S}}\omega_\sigma(t,x)\cdot d\mathcal{S}(x)=\mathbb{E}\bigg[\int_{\xi_{\sigma t,s}(\mathcal{S})}\omega_\sigma(s,y)\cdot d\mathcal{S}(y)\bigg],$$

where $\mathcal{C}$ and $\mathcal{S}$ are respectively any 1-dimensional closed curve and any 2-dimensional surface in $\mathbb{T}^3$.

3.4.2. The case of a closed manifold M

On a n-dimensional smooth closed manifold M, the non-ideal two-fluid equations, written in Eulerian form, read

$$\partial_t\omega_{i,e}+\pounds_{u_{i,e}}\omega_{i,e}-\mu\Delta_{\mathrm{H}}\omega_{i,e}=0, \tag{87}$$
$$u_i^\flat=p_i-\frac{e}{m_i}A,\qquad u_e^\flat=p_e+\frac{e}{m_e}A, \tag{88}$$
$$dp_{i,e}=\omega_{i,e},\quad d^*p_{i,e}=0, \tag{89}$$
$$\big(\lambda^2-(-1)^{n+1}\Delta_{\mathrm{H}}\big)A=\mu_0ne(p_i-p_e),\quad d^*A=0, \tag{90}$$

where $u^\flat_{i,e}$ are the velocity 1-forms associated with the velocity vector fields $u_{i,e}$, and where $\omega_{i,e}$ are exact vorticity 2-forms deriving from the canonical momenta 1-forms $p_{i,e}$. Eqs. (87)–(90) have to be supplemented with the initial conditions $\omega_{i,e}(0,x)=\omega_{i,e\,0}(x)$, subjected to the constraints $d\omega_{i,e\,0}=0$. The power $n+1$

(n being the dimension of M) in front of the Hodge–De Rham Laplacian comes from the rewriting of the Maxwell–Ampère equation (without displacement current) with the exterior coderivative d^* (see Notation 1), i.e., $\mu_0 j = *d*B = (-1)^{n+1} d^* B$, with the current 1-form $j = J^\flat$. Indeed, in the one hand we have $j = ne(u_i^\flat - u_e^\flat) = ne(p_i - p_e) - \lambda^2 A/\mu_0$ and in the other hand we have $(-1)^{n+1} d^* B = (-1)^{n+1} d^* dA = -(-1)^{n+1} \Delta_{\mathrm{H}} A$, because $d^* A = 0$. Following arguments of Section 3.2.2, from the combination of Theorem 1 and Lemma 1, we obtain the following Lagrangian formulation for the non-ideal two-fluid model (87)–(90) on a n-dimensional smooth closed manifold M: for all s such that $0 \le s \le t \le T$, for all $x \in M$, for $\sigma \in \{i, e\}$, and for all $f \in \mathscr{C}^3(M)$,

$$
\begin{aligned}
&f(\xi_{\sigma t,s}(x)) = f(x) - \int_s^t \mathrm{d}r\, u_\sigma(r) f(\xi_{\sigma t,r}(x)) - \sqrt{2\mu} \sum_{\ell=1}^m \int_s^t \mathcal{P}_\ell f(\xi_{\sigma t,r}(x)) \circ \hat{\mathrm{d}} W^\ell(r), \\
&\omega_\sigma(t,x) = \mathbb{E}\big[d(p_{\sigma i}(s) \circ \xi_{\sigma t,s}(x)) \wedge d\xi^i_{\sigma t,s}(x)\big], \\
&dp_\sigma = \omega_\sigma, \quad d^* p_\sigma = 0, \\
&\big(\lambda^2 - (-1)^{n+1} \Delta_{\mathrm{H}}\big) A = \mu_0 ne(p_i - p_e), \quad d^* A = 0, \\
&u_i^\flat = p_i - \frac{e}{m_i} A, \qquad u_e^\flat = p_e + \frac{e}{m_e} A,
\end{aligned}
$$

with the initial conditions $\omega_{i,e}(0,x) = \omega_{i,e\,0}(x)$, subjected to the constraints $d\omega_{i,e\,0} = 0$. Applying Corollary 1, we obtain the following statistical Kelvin–Helmholtz invariants,

$$
\oint_{\partial c} p_{\sigma t} = \mathbb{E}\bigg[\oint_{\xi_{\sigma t,s}(\partial c)} p_{\sigma s}\bigg], \quad \text{and} \quad \int_c \omega_{\sigma t} = \mathbb{E}\bigg[\int_{\xi_{\sigma t,s}(c)} \omega_{\sigma s}\bigg], \tag{91}
$$

for all s such that $0 \le s \le t \le T$, and for any 2-chain c of M.

Remark 10. In the case where $\nu_{\sigma_1} \neq \nu_{\sigma_2} = \mu$, with $(\sigma_1 = i, \sigma_2 = e)$ or $(\sigma_1 = e, \sigma_2 = i)$, we only have a Lagrangian formulation for the species σ_2 coupled with an Eulerian formulation for the species σ_1. Of course, the statistical Kelvin–Helmholtz invariants, such as (91), only hold for the species σ_2.

Remark 11. By forcing ions to be a uniform fixed background (i.e., $u_i = \tilde{\pi}_i = 0$, and $n_i =$ constant) in the above two-fluid model, we obtain the famous inertial e-MHD equations [29,33,79,80,93,98,130,158], which then aslo enjoy a stochastic Lagrangian formulation easily deductible from above.

4. Second application: analysis of the stochastic-Lagrangian incompressible extended MHD equations

The local-in-time well-posedness with smooth initial data and the global-in-time well-posedness with small initial data for the NI-IXMHD equations on $\mathbb{R}^3$ have been studied in [71] by using the Eulerian setting (51)–(53). Here, we perform the analysis of the stochastic-Lagrangian incompressible extended MHD equations (SL-IXMHD) given by (56)–(59), in the same spirit as it is done for the incompressible Navier–Stokes equations in [94,95,168]. Using this new stochastic Lagrangian formulation, we give in Section 4.1 a self-contained and an alternative (Lagrangian) proof of the local-in-time existence of solutions in Hölder spaces for the NI-IXMHD equations. Since our estimates are independent of the resistivity parameters, we can consider the non-resistive limit and we show in Section 4.2 that the solutions of the SL-IXMHD equations (with periodic boundary conditions) converge to the solutions of the ideal IXMHD as the resistivity parameters tend to zero, at a rate of one-half with respect to the resistivity parameters and the time variable. For large resistivity or small initial data, we give a new proof of the global-in-time existence of the NI-IXMHD equations in Section 4.3.

We start by setting our functional framework, namely the Hölder spaces equipped with non-dimensional norms (see, e.g., [76]). Non-dimensional norms are here useful to make appear magnetic Reynolds numbers.

We consider the 3-dimensional spatially periodic flat torus $\mathbb{T}^3_L$ of period L. For $k \in \mathbb{N}$ and $0 < \gamma < 1$, we define the non-dimensional Hölder norms and semi-norms on $\mathbb{T}^3_L$, by

$$\begin{aligned} |f|_\gamma &= |f|_{\mathscr{C}^{0,\gamma}(\mathbb{T}^3_L)} = \sup_{x \neq y \in \mathbb{T}^3_L} L^\gamma \frac{|f(x)-f(y)|}{|x-y|^\gamma}, \\ \|f\|_k &= \|f\|_{\mathscr{C}^k(\mathbb{T}^3_L)} = \sum_{|m|\le k} \sup_{x\in\mathbb{T}^3_L} L^{|m|}|\partial^m f(x)|, \\ |f|_{k,\gamma} &= |f|_{\mathscr{C}^{k,\gamma}(\mathbb{T}^3_L)} = \sum_{|m|=k} L^k |\partial^m f|_\gamma, \\ \|f\|_{k,\gamma} &= \|f\|_{\mathscr{C}^{k,\gamma}(\mathbb{T}^3_L)} = \|f\|_k + |f|_{k,\gamma}, \end{aligned}$$

where ∂^m stands for the derivatives with respect to the multi-index m. The space $\mathscr{C}^k(\mathbb{T}^3_L)$, of all k-times continuously differentiable periodic functions on $\mathbb{T}^3_L$, is equipped with the norm $\|f\|_{\mathscr{C}^k(\mathbb{T}^3_L)}$. The Hölder space $\mathscr{C}^{k,\gamma}(\mathbb{T}^3_L)$, of all periodic functions on $\mathbb{T}^3_L$ whose any derivatives up to the order k are γ-Hölder continuous, is equipped with the norm $\|f\|_{\mathscr{C}^{k,\gamma}(\mathbb{T}^3_L)}$.

In order to perform the analysis of the system (56)–(59), we take $s = 0$ in (56)–(57), but of course it works also for any s, such that $0 \le s < t$, provided that we know the initial condition at time s. Therefore, knowing the initial condition $A_{\pm 0}$ and using $W(0) = 0$, we aim at solving the following set of equations,

$$\xi_{\pm t,0}(x) = x - \int_0^t \mathrm{d}r\, v_\pm(r, \xi_{\pm t,r}(x)) - \sqrt{2\eta_\pm}W(t), \tag{92}$$

$$B_\pm(t,x) = \sum_{i=1,2,3} \mathbb{E}\big[\nabla(A^i_{\pm 0}\circ\xi_{\pm t,0}(x)) \times \nabla\xi^i_{\pm t,0}(x)\big], \tag{93}$$

$$\begin{aligned} v_\pm(t,x) &= \nabla\times(-\Delta)^{-1}\left(\frac{B_+(t,x)-B_-(t,x)}{\kappa_+-\kappa_-}\right) \\ &\quad - \kappa_\mp\nabla\times(1-d_e^2\Delta)^{-1}\left(\frac{\kappa_+B_-(t,x)-\kappa_-B_+(t,x)}{\kappa_+-\kappa_-}\right). \end{aligned} \tag{94}$$

4.1. Local well-posedness of the SL-IXMHD equations

Here, we state the theorem of the local-in-time well-posedness of the system (92)–(94) in Hölder spaces. We recall that, when $s \le t$, the notation $\xi_{\pm t,s}$ stands for the backward stochastic flows, which start at time t and end at time s. We prefer this notation to the notation ξ^{-1}, because it is less cumbersome and moreover the notation $\xi_{\pm t,s}$ reminds well that the backward stochastic flow is considered as a function of two time variables. Of course, the functions $\xi_{\pm t,s}$ are also functions of the spatial variable x, which is the position of the stochastic flows at time t.

Theorem 3. *Let $k \ge 2$ and $A_{\pm 0} \in \mathscr{C}^{k+1,\gamma}(\mathbb{T}^3_L;\mathbb{R}^3)$ be divergence free. Then, there exist absolute constant $\delta = \delta(k,\gamma,d_e,d_i)$ and $\kappa = \kappa(k,\gamma,d_e,d_i)$ such that for $V := \kappa(\|A_{+0}\|_{k+1,\gamma} + \|A_{-0}\|_{k+1,\gamma})$ and any time T such that $VT/L < \delta$, there exist functions $\xi_{\pm t,s} \in \mathscr{C}([0,T]\times[0,T];\mathscr{C}^{k+1,\gamma}(\mathbb{T}^3_L;\mathbb{T}^3_L))$, $v_\pm \in \mathscr{C}([0,T];\mathscr{C}^{k+1,\gamma}(\mathbb{T}^3_L;\mathbb{R}^3))$, and $B_\pm \in \mathscr{C}([0,T];\mathscr{C}^{k,\gamma}(\mathbb{T}^3_L;\mathbb{R}^3))$ such that $(\xi_{\pm t,s}, v_\pm, B_\pm)$ satisfy the system (92)–(94) with $\sup_{t\in[0,T]}\|v_\pm(t)\|_{k+1,\gamma} \le V$.*

Remark 12. We can also obtain the same type of result as Theorem 3 for the two-fluid model (82)–(86) and the inertial e-MHD equations (see Remark 11) because the structure and properties of these models are very close to the system (92)–(94). Therefore, their proofs will be quite similar. Indeed, each system is constituted by the coupling of three equations, which are a stochastic differential equation, a Cauchy invariants equation and a second-order elliptic equation giving the drift velocity of the stochastic flow from the Cauchy invariants. In the three systems mentioned above the stochastic differential equations and the Cauchy invariants equations are the same. Only the second-order elliptic equations differ, but their elliptic regularity estimates are the same, which is the only thing that matters.

Proof of Theorem 3. We first introduce the functional spaces Υ and Φ defined as follows: for any $k \geq 2$,

$$\begin{aligned} \Upsilon \;=\; & \Upsilon^{k,\gamma}_{T,V} \\ := & \left\{ \, \upsilon \in \mathscr{C}([0,T];\mathscr{C}^{k+1,\gamma}(\mathbb{T}^3_L;\mathbb{R}^3)) \;\middle|\; \forall\, t \in [0,T],\ \|\upsilon(t)\|_{k+1,\gamma} \leq V,\ \ \nabla\cdot\upsilon = 0,\ \ \upsilon(0) = \upsilon_0 \, \right\} \end{aligned} \tag{95}$$

and

$$\begin{aligned} \Phi \;=\; & \Phi^{k,\gamma}_{T,\Lambda} \\ := & \left\{ \varphi \in \mathscr{C}([0,T]\times[0,T];\mathscr{C}^{k+1,\gamma}(\mathbb{T}^3_L;\mathbb{T}^3_L)) \;\middle|\; \forall\,(s,t) \text{ s.t. } 0 \leq s \leq t \leq T,\ \|\nabla\varphi_{t,s}\|_{k,\gamma} \leq \Lambda \right\}, \end{aligned} \tag{96}$$

where Λ is a pure numerical constant. We introduce the map

$$\begin{array}{rccl} \mathcal{M} \quad : & \Upsilon \times \Upsilon & \longrightarrow & \Upsilon \times \Upsilon \\ & \mathfrak{u} = (\upsilon_+, \upsilon_-) & \longrightarrow & \tilde{\mathfrak{u}} = (\tilde{\upsilon}_+, \tilde{\upsilon}_-) = \mathcal{M}(\upsilon_+, \upsilon_-) = \mathcal{M}(\mathfrak{u}) \end{array} \tag{97}$$

which is defined by the following iterative procedure. From any given $\upsilon_\pm \in \Upsilon$, we construct $\xi_{\pm t,s}(x)$ as the unique solution of the following backward stochastic differential equation,

$$\hat{\mathrm{d}}\xi_{\pm t,s}(x) = \upsilon_\pm(s, \xi_{\pm t,s}(x))\mathrm{d}s + \sqrt{2\eta_\pm}\,\hat{\mathrm{d}}W(s), \quad 0 \leq s \leq t \leq T. \tag{98}$$

Since the Brownian process W is independent of the space variable x, the spatial derivative of (98) gives the following ordinary differential equation for the jacobian matrix $D\xi_{\pm t,s}(x)$,

$$\frac{\mathrm{d}}{\mathrm{d}s} D\xi_{\pm t,s}(x) = D\upsilon_\pm(s, \xi_{\pm t,s}(x)) D\xi_{\pm t,s}(x), \qquad 0 \leq s \leq t \leq T, \tag{99}$$

with $\xi_{\pm t,t}(x) := x$. Solving (99) backward in time, we obtain $D\xi_{\pm t,0}(x)$. From $\xi_{\pm t,0}(x)$, $D\xi_{\pm t,0}(x)$, and the initial condition $A_{\pm 0}$, we obtain

$$B_\pm(t,x) = \sum_{i=1,2,3} \mathbb{E}\big[\nabla(A^i_{\pm 0} \circ \xi_{\pm t,0}(x)) \times \nabla\xi^i_{\pm t,0}(x)\big]. \tag{100}$$

Finally, using (100), we define $\tilde{\upsilon}_\pm$ as

$$\begin{aligned} \tilde{\upsilon}_\pm \;:=\; & \mathcal{F}_\pm(B_+, B_-) \\ := & \nabla \times (-\Delta)^{-1}\left(\frac{B_+ - B_-}{\kappa_+ - \kappa_-}\right) \;-\; \kappa_\mp \nabla \times (1 - d_e^2\Delta)^{-1}\left(\frac{\kappa_+ B_- - \kappa_- B_+}{\kappa_+ - \kappa_-}\right). \end{aligned} \tag{101}$$

The idea of the proof is to use the Picard iterative process defined as above and the Banach fixed-point theorem in $\Upsilon^2 := \Upsilon \times \Upsilon$ for the map $\mathcal{M} : \Upsilon^2 \to \Upsilon^2$, where the space Υ^2 is equipped with the weaker norm

$$\|\mathfrak{u}\|_{\Upsilon^2} := \sup_{t\in[0,T]} \|\mathfrak{u}(t)\|_{k,\gamma} := \sup_{t\in[0,T]} \big(\|\upsilon_+(t)\|_{k,\gamma} + \|\upsilon_-(t)\|_{k,\gamma}\big). \tag{102}$$

To verify the hypotheses of the Banach fixed-point theorem, we need to show some basic a priori estimates given in the following lemma.

Lemma 2. *Let $k \geq 2$ and $\upsilon_\pm \in \Upsilon$. Then, there exists a constant $C_\flat = C_\flat(k,\gamma,VT/L)$ non-decreasing in the third argument VT/L, such that*

$$\|D\xi_{\pm t,s}\|_{k,\gamma} \leq \exp\left(C_\flat \frac{V}{L}|t-s|\right), \quad \forall\,(s,t) \ \text{ s.t. } 0 \leq s \leq t \leq T, \tag{103}$$

and

$$\|D\varphi_{\pm t,s}\|_{k,\gamma} \leq C_\flat \frac{V}{L}|t-s| \exp\left(C_\flat \frac{V}{L}|t-s|\right), \quad \forall\,(s,t) \ \text{ s.t. } 0 \leq s \leq t \leq T, \tag{104}$$

where $\varphi_{\pm t,s}(x) := \xi_{\pm t,s}(x) - \mathrm{Id}(x) = \xi_{\pm t,s}(x) - x$.

Proof. Integrating (98) backward in time we obtain

$$\xi_{\pm t,s}(x) = x - \int_s^t \mathrm{d}r\, v_\pm(r, \xi_{\pm t,r}(x)) + \sqrt{2\eta_\pm}(W(s) - W(t)). \tag{105}$$

Since the Brownian process W is independent of the space variable x, the spatial derivative of (105) gives

$$D\xi_{\pm t,s}(x) = I - \int_s^t \mathrm{d}r\, Dv_\pm(r, \xi_{\pm t,r}(x)) D\xi_{\pm t,r}(x). \tag{106}$$

Since $v_\pm \in \Upsilon$, using a Gronwall lemma, we obtain from (106),

$$\|D\xi_{\pm t,s}\|_{L^\infty(\mathbb{T}_L^3)} \leq \exp\biggl(\int_s^t \mathrm{d}r \sup_{r\in[0,T]} \|Dv_\pm(r)\|_{L^\infty(\mathbb{T}_L^3)}\biggr) \leq \exp\biggl(\frac{V}{L}|t-s|\biggr).$$

By spatial differentiation of (105), we obtain

$$D\varphi_{\pm t,s}(x) = D\xi_{\pm t,s}(x) - I = -\int_s^t \mathrm{d}r\, Dv_\pm(r, \xi_{\pm t,r}(x)) D\varphi_{\pm t,r}(x) - \int_s^t \mathrm{d}r\, Dv_\pm(r, \xi_{\pm t,r}(x)) \tag{107}$$

which, by using a Gronwall lemma, leads to the following estimate

$$\begin{aligned}\|D\varphi_{\pm t,s}\|_{L^\infty(\mathbb{T}_L^3)} &\leq \int_s^t \mathrm{d}r\, \|Dv_\pm(r)\|_{L^\infty(\mathbb{T}_L^3)} + \int_s^t \mathrm{d}r\, \|Dv_\pm(r)\|_{L^\infty(\mathbb{T}_L^3)} \|D\varphi_{\pm t,r}\|_{L^\infty(\mathbb{T}_L^3)} \\ &\leq \biggl(\int_s^t \mathrm{d}r\, \|Dv_\pm(r)\|_{L^\infty(\mathbb{T}_L^3)}\biggr) \exp\biggl(\int_s^t \mathrm{d}r\, \|Dv_\pm(r)\|_{L^\infty(\mathbb{T}_L^3)}\biggr) \\ &\leq \frac{V}{L}|t-s| \exp\biggl(\frac{V}{L}|t-s|\biggr).\end{aligned}$$

Since the space $\mathscr{C}^{k,\gamma}$ is an algebra for the multiplication operation with an algebra constant $C_a = C_a(k,\gamma)$, taking the $\mathscr{C}^{k,\gamma}$-norm of (106) we obtain

$$\|D\xi_{\pm t,s}\|_{k,\gamma} \leq 1 + C_a \int_s^t \mathrm{d}r\, \|Dv_\pm(r, \xi_{\pm t,r})\|_{k,\gamma} \|D\xi_{\pm t,r}\|_{k,\gamma}. \tag{108}$$

Using the following standard inequality (see, e.g., [94]),

$$\|f \circ g\|_{k,\gamma} \leq C(k,\gamma) \|f\|_{k,\gamma} (1 + \|Dg\|_{k-1,\gamma})^{k+\gamma}, \tag{109}$$

estimate (108) becomes

$$\|D\xi_{\pm t,s}\|_{k,\gamma} \leq 1 + C(k,\gamma) \int_s^t \mathrm{d}r\, \|Dv_\pm(r)\|_{k,\gamma} (1 + \|D\xi_{\pm t,r}\|_{k-1,\gamma})^{k+\gamma} \|D\xi_{\pm t,r}\|_{k,\gamma}. \tag{110}$$

We now proceed by induction on the regularity index k. The induction hypothesis H_k is that there exists a constant $C_\flat(k,\gamma,VT/L)$, non decreasing in the third argument VT/L, such that

$$\|D\xi_{\pm t,s}\|_{k-1,\gamma} \leq \exp\biggl(C_\flat \frac{V}{L}|t-s|\biggr). \tag{111}$$

Using H_k we obtain

$$(1 + \|D\xi_{\pm t,r}\|_{k-1,\gamma})^{k+\gamma} \leq \biggl(1 + \exp\biggl(C_\flat \frac{VT}{L}\biggr)\biggr)^{k+\gamma} \leq C_\flat(k,\gamma,VT/L),$$

which, by using a Gronwall lemma, allows us to obtain from (110),

$$\begin{aligned}\|D\xi_{\pm t,s}\|_{k,\gamma} &\leq 1 + C_\flat \int_s^t \mathrm{d}r\, \|Dv_\pm(r)\|_{k,\gamma} \|D\xi_{\pm t,r}\|_{k,\gamma} \\ &\leq \exp\biggl(C_\flat \int_s^t \mathrm{d}r \sup_{r\in[0,T]} \|Dv_\pm(r)\|_{k,\gamma}\biggr) \\ &\leq \exp\biggl(C_\flat \frac{V}{L}|t-s|\biggr).\end{aligned}$$

This ends the proof of the first estimate of Lemma 2. Using the algebra property of the space $\mathscr{C}^{k,\gamma}$ and inequality (109), we obtain from the $\mathscr{C}^{k,\gamma}$-norm of (107),

$$\begin{aligned}
\|D\varphi_{\pm t,s}\|_{k,\gamma} &\leq \int_s^t \mathrm{d}r\, \|Dv_\pm(r,\xi_{\pm t,r})\|_{k,\gamma} + C_a \int_s^t \mathrm{d}r\, \|Dv_\pm(r,\xi_{\pm t,r})\|_{k,\gamma} \|D\varphi_{\pm t,r}\|_{k,\gamma} \\
&\leq C(k,\gamma) \int_s^t \mathrm{d}r\, \|Dv_\pm(r)\|_{k,\gamma} (1 + \|D\xi_{\pm t,r}\|_{k-1,\gamma})^{k+\gamma} \\
&\quad + C(k,\gamma) \int_s^t \mathrm{d}r\, \|Dv_\pm(r)\|_{k,\gamma} (1 + \|D\xi_{\pm t,r}\|_{k-1,\gamma})^{k+\gamma} \|D\varphi_{\pm t,r}\|_{k,\gamma}.
\end{aligned}$$

Using (111) and a Gronwall lemma we obtain from the previous estimate

$$\begin{aligned}
\|D\varphi_{\pm t,s}\|_{k,\gamma} &\leq C(k,\gamma) \left(1 + \exp\left(C_\flat \frac{VT}{L}\right)\right)^{k+\gamma} \frac{V}{L} |t-s| \\
&\quad + C(k,\gamma) \int_s^t \mathrm{d}r\, \|Dv_\pm(r)\|_{k,\gamma} \left(1 + \exp\left(C_\flat \frac{VT}{L}\right)\right)^{k+\gamma} \|D\varphi_{\pm t,r}\|_{k,\gamma} \\
&\leq C_\flat \frac{V}{L} |t-s| + C_\flat \int_s^t \mathrm{d}r\, \|Dv_\pm(r)\|_{k,\gamma} \|D\varphi_{\pm t,r}\|_{k,\gamma} \\
&\leq C_\flat \frac{V}{L} |t-s| \exp\left(C_\flat \frac{V}{L} |t-s|\right),
\end{aligned}$$

which ends the proof of the second estimate of Lemma 2.

We continue by stating a lemma which gives a stability or a comparison result between two solutions of the backward stochastic differential Eq. (98).

Lemma 3. *Let $k \geq 2$ and $v_\pm, \tilde{v}_\pm \in \Upsilon$. Let $\xi_{\pm t,s}$ and $\tilde{\xi}_{\pm t,s}$ be the unique solutions of the backward stochastic differential Eq. (98) associated with the velocities $v_\pm$ and $\tilde{v}_\pm$ respectively. Then, there exists a constant $C_\flat = C_\flat(k,\gamma,VT/L)$, non decreasing in the third argument VT/L, such that*

$$\|\xi_{\pm t,s} - \tilde{\xi}_{\pm t,s}\|_{k,\gamma} \leq C_\flat \exp\left(C_\flat \frac{V}{L} |t-s|\right) \int_s^t \mathrm{d}r\, \|v_\pm(r) - \tilde{v}_\pm(r)\|_{k,\gamma},$$

for all (s,t) such that $0 \leq s \leq t \leq T$.

Proof. Obviously the two solutions $\xi_{\pm t,s}$ and $\tilde{\xi}_{\pm t,s}$ of the backward stochastic differential Eq. (98) satisfy

$$\xi_{\pm t,s} - \tilde{\xi}_{\pm t,s} = -\int_s^t \mathrm{d}r\, \big(v_\pm(r,\xi_{\pm t,r}(x)) - \tilde{v}_\pm(r,\tilde{\xi}_{\pm t,r}(x))\big). \tag{112}$$

Using the following standard inequality (see, e.g., [94]),

$$\begin{aligned}
\|f\circ g - \tilde{f}\circ\tilde{g}\|_{k,\gamma} &\leq C(k,\gamma)(1 + \|Dg\|_{k-1,\gamma} + \|D\tilde{g}\|_{k-1,\gamma})^{k+1} \\
&\quad \Big[\|f-\tilde{f}\|_{k,\gamma} + \min\big\{\|Df\|_{k,\gamma}\,,\, \|D\tilde{f}\|_{k,\gamma}\big\}\|g-\tilde{g}\|_{k,\gamma}\Big],
\end{aligned} \tag{113}$$

estimate (111) and a Gronwall lemma, we obtain from (112),

$$\begin{aligned}
\|\xi_{\pm t,s} - \tilde{\xi}_{\pm t,s}\|_{k,\gamma} &\leq C(k,\gamma) \int_s^t \mathrm{d}r\, (1 + \|D\xi_{\pm t,r}\|_{k-1,\gamma} + \|D\tilde{\xi}_{\pm t,r}\|_{k-1,\gamma})^{k+1} \\
&\quad \Big[\|v_\pm(r) - \tilde{v}_\pm(r)\|_{k,\gamma} + \min\big\{\|Dv_\pm(r)\|_{k,\gamma}\,,\, \|D\tilde{v}_\pm(r)\|_{k,\gamma}\big\}\|\xi_{\pm t,r} - \tilde{\xi}_{\pm t,r}\|_{k,\gamma}\Big] \\
&\leq C(k,\gamma) \int_s^t \mathrm{d}r\, \left(1 + \exp\left(C_\flat \frac{V}{L}|t-r|\right)\right)^{k+\gamma}
\end{aligned}$$

$$
\begin{aligned}
&\left[\|v_\pm(r)-\tilde v_\pm(r)\|_{k,\gamma}+\frac{V}{L}\|\xi_{\pm t,r}-\tilde\xi_{\pm t,r}\|_{k,\gamma}\right]\\
&\le C_\flat(k,\gamma,VT/L)\int_s^t \mathrm{d}r\left[\|v_\pm(r)-\tilde v_\pm(r)\|_{k,\gamma}+\frac{V}{L}\|\xi_{\pm t,r}-\tilde\xi_{\pm t,r}\|_{k,\gamma}\right]\\
&\le C_\flat \exp\left(C_\flat\frac{V}{L}|t-s|\right)\int_s^t \mathrm{d}r\,\|v_\pm(r)-\tilde v_\pm(r)\|_{k,\gamma},
\end{aligned}
$$

which ends the proof of Lemma 3.

In order to prove the existence of a fixed point for the map $\mathcal{M}=\Upsilon^2\to\Upsilon^2$, with the space Υ defined by (95), we have to show that the application $\mathcal{M}$ maps Υ^2 into itself and that the application $\mathcal{M}$ is a contraction for the norm $\|\cdot\|_{\Upsilon^2}$ defined by (102). We first show that $\mathcal{M}$ maps Υ^2 into itself. Let $C_{\mathcal{F}}$ be the continuity constant of the integro-differential linear operator $\mathcal{F}:=(\mathcal{F}_+,\mathcal{F}_-)$ defined by (101) and which is continuous from $\mathscr{C}^{k,\gamma}(\mathbb{T}^3_L;\mathbb{R}^3)\times\mathscr{C}^{k,\gamma}(\mathbb{T}^3_L;\mathbb{R}^3)$ into $\mathscr{C}^{k+1,\gamma}(\mathbb{T}^3_L;\mathbb{R}^3)\times\mathscr{C}^{k+1,\gamma}(\mathbb{T}^3_L;\mathbb{R}^3)$. This results from the regularity estimates of the linear elliptic operator $\mathcal{F}$ (see, e.g., [76]). We note that the constant $C_{\mathcal{F}}=C_{\mathcal{F}}(k,\gamma,d_e,d_i)$ depends on k, γ, d_e and d_i. Using elliptic regularity estimates, the algebra property of the space $\mathscr{C}^{k,\gamma}$ and inequality (109), we obtain

$$
\begin{aligned}
\|\mathcal{M}(\mathfrak{u}(t))\|_{k+1,\gamma} &= \|\mathcal{M}(v_+(t),v_-(t))\|_{k+1,\gamma}\\
&:= \|\mathcal{F}_+(B_+(t),B_-(t))\|_{k+1,\gamma}+\|\mathcal{F}_-(B_+(t),B_-(t))\|_{k+1,\gamma}\\
&\le C_{\mathcal{F}}L\big(\|B_+(t)\|_{k,\gamma}+\|B_-(t)\|_{k,\gamma}\big)\\
&\le C_{\mathcal{F}}L\sum_{i=1,2,3}\Big(\big\|\mathbb{E}\big[\nabla(A^i_{+0}\circ\xi_{+t,0})\times\nabla\xi^i_{+t,0}\big]\big\|_{k,\gamma}\\
&\qquad\qquad+\big\|\mathbb{E}\big[\nabla(A^i_{-0}\circ\xi_{-t,0})\times\nabla\xi^i_{-t,0}\big]\big\|_{k,\gamma}\Big)\\
&\le C_{\mathcal{F}}C_a^2L\sup_{\Omega}\Big\{\|DA_{+0}\circ\xi_{+t,0}\|_{k,\gamma}\|D\xi_{+t,0}\|^2_{k,\gamma}+\|DA_{-0}\circ\xi_{-t,0}\|_{k,\gamma}\|D\xi_{-t,0}\|^2_{k,\gamma}\Big\}\\
&\le C_{\mathcal{F}}C_a^2L\sup_{\Omega}\Big\{\|A_{+0}\|_{k+1,\gamma}(1+\|D\xi_{+t,0}\|_{k-1,\gamma})^{k+\gamma}\|D\xi_{+t,0}\|^2_{k,\gamma}\\
&\qquad\qquad+\|A_{-0}\|_{k+1,\gamma}(1+\|D\xi_{-t,0}\|_{k-1,\gamma})^{k+\gamma}\|D\xi_{-t,0}\|^2_{k,\gamma}\Big\}\\
&\le C_{\mathcal{F}}C_a^2L\sup_{\Omega}\Big\{\|A_{+0}\|_{k+1,\gamma}(2+\|D\varphi_{+t,0}\|_{k,\gamma})^{k+\gamma+2}\\
&\qquad\qquad+\|A_{-0}\|_{k+1,\gamma}(2+\|D\varphi_{-t,0}\|_{k,\gamma})^{k+\gamma+2}\Big\},
\end{aligned}\tag{114}
$$

where Ω is the probability space on which the stochastic processes are defined. Using Lemma 2, we choose T small enough so that $\|D\varphi_{\pm t,0}\|_{k,\gamma}\le\Lambda$, for all $t\in[0,T]$, i.e., $\varphi_{\pm t,0}\in\Phi$, where the space Φ has been defined by (96). Therefore, we obtain from (114),

$$
\|\mathcal{M}(\mathfrak{u}(t))\|_{k+1,\gamma}\le C_{\mathcal{F}}C_a^2(2+\Lambda)^{k+\gamma+2}\big(\|A_{+0}\|_{k+1,\gamma}+\|A_{-0}\|_{k+1,\gamma}\big).
$$

Setting

$$
\kappa=\kappa(k,\gamma,d_e,d_i):=C_{\mathcal{F}}C_a^2(2+\Lambda)^{k+\gamma+2},\qquad\text{and}\qquad V:=\kappa\big(\|A_{+0}\|_{k+1,\gamma}+\|A_{-0}\|_{k+1,\gamma}\big),
$$

we obtain $\sup_{t\in[0,T]}\|\mathcal{M}(\mathfrak{u}(t))\|_{k+1,\gamma}\le V$ and then $\mathcal{M}$ maps Υ^2 into itself. We now show that the map $\mathcal{M}$ is a contraction for the norm $\|\cdot\|_{\Upsilon^2}$ defined by (102). Using the algebra property of the space $\mathscr{C}^{k,\gamma}$ and the continuity of the operator $\mathcal{F}$ from $\mathscr{C}^{k,\gamma}(\mathbb{T}^3_L;\mathbb{R}^3)\times\mathscr{C}^{k,\gamma}(\mathbb{T}^3_L;\mathbb{R}^3)$ into $\mathscr{C}^{k+1,\gamma}(\mathbb{T}^3_L;\mathbb{R}^3)\times\mathscr{C}^{k+1,\gamma}(\mathbb{T}^3_L;\mathbb{R}^3)$, we obtain

$$
\|\mathcal{M}(\mathfrak{u}(t))-\mathcal{M}(\tilde{\mathfrak{u}}(t))\|_{k,\gamma} := \|\mathcal{F}_+(B_+(t)-\tilde B_+(t),B_-(t)-\tilde B_-(t))\|_{k,\gamma}+
$$

$$
\begin{aligned}
&\|\mathcal{F}_-(B_+(t)-\tilde{B}_+(t),B_-(t)-\tilde{B}_-(t))\|_{k,\gamma}\\
&\le C_{\mathcal{F}}L\big(\|B_+(t)-\tilde{B}_+(t)\|_{k-1,\gamma}+\|B_-(t)-\tilde{B}_-(t)\|_{k-1,\gamma}\big)\\
&\le C_{\mathcal{F}}L\sum_{i=1,2,3}\Big(\big\|\mathbb{E}\big[\nabla(A^i_{+0}\circ\xi_{+t,0})\times\nabla\xi^i_{+t,0}-\nabla(\tilde{A}^i_{+0}\circ\tilde{\xi}_{+t,0})\times\nabla\tilde{\xi}^i_{+t,0}\big]\big\|_{k-1,\gamma}\\
&\qquad+\ \big\|\mathbb{E}\big[\nabla(A^i_{-0}\circ\xi_{-t,0})\times\nabla\xi^i_{-t,0}-\nabla(\tilde{A}^i_{-0}\circ\tilde{\xi}_{-t,0})\times\nabla\tilde{\xi}^i_{-t,0}\big]\big\|_{k-1,\gamma}\Big)\\
&\le C_{\mathcal{F}}C_a^2L\sup_{\Omega}\Big\{\|DA_{+0}\circ\xi_{+t,0}-D\tilde{A}_{+0}\circ\tilde{\xi}_{+t,0}\|_{k-1,\gamma}\|D\xi_{+t,0}\|^2_{k-1,\gamma}+\\
&\qquad\|D\tilde{A}_{+0}\circ\tilde{\xi}_{+t,0}\|_{k-1,\gamma}\|D\xi_{+t,0}-D\tilde{\xi}_{+t,0}\|_{k-1,\gamma}(\|D\xi_{+t,0}\|_{k-1,\gamma}+\|D\tilde{\xi}_{+t,0}\|_{k-1,\gamma})+\\
&\qquad\|DA_{-0}\circ\xi_{-t,0}-D\tilde{A}_{-0}\circ\tilde{\xi}_{-t,0}\|_{k-1,\gamma}\|D\xi_{-t,0}\|^2_{k-1,\gamma}+\\
&\qquad\|D\tilde{A}_{-0}\circ\tilde{\xi}_{-t,0}\|_{k-1,\gamma}\|D\xi_{-t,0}-D\tilde{\xi}_{-t,0}\|_{k-1,\gamma}(\|D\xi_{-t,0}\|_{k-1,\gamma}+\|D\tilde{\xi}_{-t,0}\|_{k-1,\gamma})\Big\}.
\end{aligned}
$$

Using inequalities (109) and (113), the previous estimate becomes

$$
\begin{aligned}
&\|\mathcal{M}(\mathfrak{u}(t))-\mathcal{M}(\tilde{\mathfrak{u}}(t))\|_{k,\gamma}\\
&\le C_{\mathcal{F}}C_a^2L\sup_{\Omega}\Big\{(1+\|D\xi_{+t,0}\|_{k-2,\gamma}+\|D\tilde{\xi}_{+t,0}\|_{k-2,\gamma})^k\|D\xi_{+t,0}\|^2_{k-1,\gamma}\\
&\Big[\|DA_{+0}-D\tilde{A}_{+0}\|_{k-1,\gamma}\ +\ \min\big(\|D^2A_{+0}\|_{k-1,\gamma}\,,\,\|D^2\tilde{A}_{+0}\|_{k-1,\gamma}\big)\|\xi_{+t,0}-\tilde{\xi}_{+t,0}\|_{k-1,\gamma}\Big]+\\
&\|D\tilde{A}_{+0}\|_{k-1,\gamma}(1+\|D\tilde{\xi}_{+t,0}\|_{k-2,\gamma})^{k-1+\gamma}\|D\xi_{+t,0}-D\tilde{\xi}_{+t,0}\|_{k-1,\gamma}(\|D\xi_{+t,0}\|_{k-1,\gamma}+\|D\tilde{\xi}_{+t,0}\|_{k-1,\gamma})\\
&+\ (1+\|D\xi_{-t,0}\|_{k-2,\gamma}+\|D\tilde{\xi}_{-t,0}\|_{k-2,\gamma})^k\|D\xi_{-t,0}\|^2_{k-1,\gamma}\\
&\Big[\|DA_{-0}-D\tilde{A}_{-0}\|_{k-1,\gamma}\ +\ \min\big(\|D^2A_{-0}\|_{k-1,\gamma}\,,\,\|D^2\tilde{A}_{-0}\|_{k-1,\gamma}\big)\|\xi_{-t,0}-\tilde{\xi}_{-t,0}\|_{k-1,\gamma}\Big]+\\
&\|D\tilde{A}_{-0}\|_{k-1,\gamma}(1+\|D\tilde{\xi}_{-t,0}\|_{k-2,\gamma})^{k-1+\gamma}\|D\xi_{-t,0}-D\tilde{\xi}_{-t,0}\|_{k-1,\gamma}(\|D\xi_{-t,0}\|_{k-1,\gamma}+\|D\tilde{\xi}_{-t,0}\|_{k-1,\gamma})\Big\}.
\end{aligned}
\tag{115}
$$

Using Lemma 2, we choose T small enough so that $\|D\varphi_{\pm t,0}\|_{k-1,\gamma}\le\varLambda$ and $\|D\tilde{\varphi}_{\pm t,0}\|_{k-1,\gamma}\le\varLambda$, for all $t\in[0,T]$, hence $\|D\xi_{\pm t,0}\|_{k-1,\gamma}\le 1+\varLambda$ and $\|D\tilde{\xi}_{\pm t,0}\|_{k-1,\gamma}\le 1+\varLambda$. Therefore, we obtain from (115),

$$
\begin{aligned}
&\|\mathcal{M}(\mathfrak{u}(t))-\mathcal{M}(\tilde{\mathfrak{u}}(t))\|_{k,\gamma}\\
&\le C_{\mathcal{F}}C_a^2L\sup_{\Omega}\Big\{(3+2\varLambda)^{k+2}\Big[L^{-1}\|A_{+0}-\tilde{A}_{+0}\|_{k,\gamma}\\
&\quad+\ L^{-2}\min\big(\|A_{+0}\|_{k+1,\gamma}\,,\,\|\tilde{A}_{+0}\|_{k+1,\gamma}\big)\|\xi_{+t,0}-\tilde{\xi}_{+t,0}\|_{k-1,\gamma}\Big]\\
&\quad+\ L^{-2}\|\tilde{A}_{+0}\|_{k,\gamma}(2+2\varLambda)^{k+\gamma}\|\xi_{+t,0}-\tilde{\xi}_{+t,0}\|_{k,\gamma}\\
&\quad+\ (3+2\varLambda)^{k+\gamma+2}\Big[L^{-1}\|A_{-0}-\tilde{A}_{-0}\|_{k,\gamma}\\
&\quad+\ L^{-2}\min\big(\|A_{-0}\|_{k+1,\gamma}\,,\,\|\tilde{A}_{-0}\|_{k+1,\gamma}\big)\|\xi_{-t,0}-\tilde{\xi}_{-t,0}\|_{k-1,\gamma}\Big]\\
&\quad+\ L^{-2}\|\tilde{A}_{-0}\|_{k,\gamma}(2+2\varLambda)^{k+\gamma}\|\xi_{-t,0}-\tilde{\xi}_{-t,0}\|_{k,\gamma}\Big\}\\
&\le C(k,\gamma,d_e,d_i,\varLambda)\sup_{\Omega}\Big\{\|A_{+0}-\tilde{A}_{+0}\|_{k,\gamma}+\|A_{-0}-\tilde{A}_{-0}\|_{k,\gamma}\\
&\quad+\ L^{-1}\max\big(\|A_{+0}\|_{k+1,\gamma}\,,\,\|A_{-0}\|_{k+1,\gamma}\,,\,\|\tilde{A}_{+0}\|_{k+1,\gamma}\,,\,\|\tilde{A}_{-0}\|_{k+1,\gamma}\big)\\
&\qquad\big(\|\xi_{+t,0}-\tilde{\xi}_{+t,0}\|_{k-1,\gamma}+\|\xi_{-t,0}-\tilde{\xi}_{-t,0}\|_{k-1,\gamma}\big)\\
&\quad+\ L^{-1}\big(\|\tilde{A}_{+0}\|_{k,\gamma}+\|\tilde{A}_{-0}\|_{k,\gamma}\big)\big[\|\xi_{+t,0}-\tilde{\xi}_{+t,0}\|_{k,\gamma}+\|\xi_{-t,0}-\tilde{\xi}_{-t,0}\|_{k,\gamma}\big]\Big\}.
\end{aligned}
$$

Using Lemma 3 and the previous estimate, we obtain

$$
\begin{aligned}
\|\mathcal{M}(\mathfrak{u}(t))-\mathcal{M}(\tilde{\mathfrak{u}}(t))\|_{k,\gamma}&\le C(k,\gamma,d_e,d_i,\varLambda)\big\{\|A_{+0}-\tilde{A}_{+0}\|_{k,\gamma}+\|A_{-0}-\tilde{A}_{-0}\|_{k,\gamma}\big\}\\
&+C_\flat\frac{V}{L}\exp\Big(C_\flat\frac{V}{L}t\Big)\int_0^t\mathrm{d}r\,\big(\|v_+(r)-\tilde{v}_+(r)\|_{k,\gamma}+\|v_-(r)-\tilde{v}_-(r)\|_{k,\gamma}\big).
\end{aligned}
\tag{116}
$$

Without loss of generality, we can assume the same initial conditions, i.e., $A_{\pm 0} = \tilde{A}_{\pm 0}$. Then, using definition (102) for the norm $\|\cdot\|_{\Upsilon^2}$, and taking the supremum in time in estimate (116) for $t \in [0, T]$, we obtain

$$\|\mathcal{M}(\mathfrak{u}) - \mathcal{M}(\tilde{\mathfrak{u}})\|_{\Upsilon^2} \leq C_\flat \frac{VT}{L} \exp\left(C_\flat \frac{VT}{L}\right) \|\mathfrak{u} - \tilde{\mathfrak{u}}\|_{\Upsilon^2}. \tag{117}$$

The map $\mathcal{M}$ is contracting in $(\Upsilon^2, \|\cdot\|_{\Upsilon^2})$ if $C_\flat VT/L \exp(C_\flat VT/L) < 1$, where $C_\flat = C_\flat(k, \gamma, d_e, d_i, VT/L)$ is a non decreasing function in its fourth argument VT/L. Using the local inversion theorem or the implicit function theorem, there exists a constant $\delta = \delta(k, \gamma, d_e, d_i)$ such that if $VT/L < \delta < 1$ then $C_\flat VT/L \exp(C_\flat VT/L) < 1$, and the map $\mathcal{M}$ is a contraction in $(\Upsilon^2, \|\cdot\|_{\Upsilon^2})$. Therefore, we obtain

$$T < \frac{\delta L}{\kappa} (\|A_{+0}\|_{k+1,\gamma} + \|A_{-0}\|_{k+1,\gamma})^{-1}.$$

As for the proof of the Banach fixed-point theorem from the Picard method of successive approximations, the existence of a fixed point for the map $\mathcal{M}$ can be constructed by successive iterations. For this, we define $\mathfrak{u}_{n+1} = \mathcal{M}(\mathfrak{u}_n)$. From the contraction property of the map $\mathcal{M}$, the sequence $\{\mathfrak{u}_n\}_{n\geq 0}$ converges strongly in $(\Upsilon^2, \|\cdot\|_{\Upsilon^2})$. Since Υ^2 is a closed and convex set, and since the sequence $\{\mathfrak{u}_n\}_{n\geq 0}$ is uniformly bounded in $\mathscr{C}^{k+1,\gamma}(\mathbb{T}^3_L; \mathbb{R}^3) \times \mathscr{C}^{k+1,\gamma}(\mathbb{T}^3_L; \mathbb{R}^3)$, this sequence has a weak limit $\bar{\mathfrak{u}} \in \Upsilon^2$. Since from the contraction property, the map $\mathcal{M}$ is continuous in $\mathscr{C}^{k,\gamma}(\mathbb{T}^3_L; \mathbb{R}^3) \times \mathscr{C}^{k,\gamma}(\mathbb{T}^3_L; \mathbb{R}^3)$, this limit $\bar{\mathfrak{u}}$ must be the fixed point of the map $\mathcal{M}$, i.e., the solution of the system (92)–(94). This ends the proof of Theorem 3. $\square$

4.2. The non-resistive limit of the SL-IXMHD equations

Here, we show that the solutions of the SL-IXMHD Eqs. (92)–(94), with periodic boundary conditions, converge to the solutions of the ideal IXMHD as the resistivity parameters $\eta := (\eta_+, \eta_-)$ tend to zero, at a rate of one-half with respect to the resistivity parameters $|\eta| = (\eta_+^2 + \eta_-^2)^{1/2}$ and the time variable. More precisely we state the following theorem.

Theorem 4. *Let $k \geq 2$ and $A_{\pm 0} \in \mathscr{C}^{k+1,\gamma}(\mathbb{T}^3_L, \mathbb{R}^3)$ be divergence free. Let V and T be as in Theorem 3, i.e., such that $VT/L < \delta(k, \gamma, d_e, d_i)$ and $V = \kappa(k, \gamma, d_e, d_i)(\|A_{+0}\|_{k+1,\gamma} + \|A_{-0}\|_{k+1,\gamma})$. For each $\eta = (\eta_+, \eta_-)$, let $(v^\eta_\pm, B^\eta_\pm)$ be the solution of the SL-IXMHD system (92)–(94) on the time interval $[0, T]$. Making T smaller if necessary, let $(v_\pm, B_\pm)$ be the solutions of the ideal incompressible extended MHD system (namely Eqs. (51)–(53) with $\eta_+ = \eta_- = 0$) with initial data $A_{\pm 0}$ and defined on the interval $[0, T]$. Then, there exist two constants $C_\sharp = C_\sharp(k, \gamma, d_e, d_i)$ and $C_* = C_*(k, \gamma, d_e, d_i)$ such that for all $t \in [0, T]$ we have*

$$\|B^\eta_\pm(t) - B_\pm(t)\|_{k,\gamma} \leq C_\sharp \frac{V}{L^2} \sqrt{|\eta| t}, \quad \text{and} \quad \|v^\eta_\pm(t) - v_\pm(t)\|_{k,\gamma} \leq C_* \frac{V}{L} \sqrt{|\eta| t}.$$

Proof. The (backward in time) Lagrangian formulation of the ideal incompressible extended MHD system reads

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s} \xi_{\pm t,s}(x) &= v_\pm(s, \xi_{\pm t,s}(x)), \quad \xi_{\pm t,t}(x) = x, \\
B_\pm(t, x) &= \sum_{i=1,2,3} \nabla(A^i_\pm(s) \circ \xi_{\pm,t,s}(x)) \times \nabla \xi^i_{\pm t,s}(x), \\
v_\pm &= \mathcal{F}_\pm(B_+, B_-), \quad \nabla \times A_\pm = B_\pm, \quad \nabla \cdot A_\pm = 0,
\end{aligned}$$

for all $x \in \mathbb{T}^3_L$ and for all (s, t) such that $0 \leq s \leq t \leq T$, and where the integro-differential operators $\mathcal{F}_\pm$ are given by (101). Let $(\xi^\eta_{\pm t,s}, v^\eta_\pm, B^\eta_\pm)$ be the unique solution of the SL-IXMHD system (92)–(94) or (56)–(59). Then, for all $x \in \mathbb{T}^3_L$ and for all (s, t) such that $0 \leq s \leq t \leq T$, we have

$$\xi^\eta_{\pm t,s}(x) - \xi_{\pm t,s}(x) = -\int_s^t \mathrm{d}r \left(v^\eta_\pm(r) \circ \xi^\eta_{\pm t,r}(x) - v_\pm(r) \circ \xi_{\pm t,r}(x)\right) + \sqrt{2\eta_\pm}(W(s) - W(t)).$$

Using this equation and following the proof of Lemma 3, we can show that there exists a constant $C_\sharp\big(k,\gamma,VT/L<\delta(k,\gamma,d_e,d_i)\big)=C_\sharp(k,\gamma,d_e,d_i)$ such that for $k\geq 2$, we have

$$\begin{aligned}&\|\xi^\eta_{\pm t,s}-\xi_{\pm t,s}\|_{k+1,\gamma}\\&\quad\leq C_\sharp\int_s^t \mathrm{d}r\,\Big(\|v^\eta_\pm(r)-v_\pm(r)\|_{k+1,\gamma}+\frac{V}{L}\|\xi^\eta_{\pm t,r}-\xi_{\pm t,r}\|_{k+1,\gamma}\Big)\;+\;\sqrt{2\eta_\pm}|W(s)-W(t)|.\end{aligned}$$

Using this estimate and a Gronwall lemma, we obtain

$$\|\xi^\eta_{\pm t,s}-\xi_{\pm t,s}\|_{k+1,\gamma}\leq C_\sharp\Big(\int_s^t \mathrm{d}r\,\|v^\eta_\pm(r)-v_\pm(r)\|_{k+1,\gamma}\;+\;\sqrt{2\eta_\pm}|W(s)-W(t)|\Big)\exp\Big(C_\sharp\frac{V}{L}|t-s|\Big).$$

Taking the expectation of the previous estimate, we obtain for all (s,t) such that $0\leq s\leq t\leq T$,

$$\mathbb{E}\|\xi^\eta_{\pm t,s}-\xi_{\pm t,s}\|_{k+1,\gamma}\leq C_\sharp\Big(\sqrt{\eta_\pm|t-s|}\;+\;\int_s^t \mathrm{d}r\,\|v^\eta_\pm(r)-v_\pm(r)\|_{k+1,\gamma}\Big)\exp\Big(C_\sharp\frac{V}{L}|t-s|\Big).\tag{118}$$

Using our Lagrangian formulation, we obtain

$$\begin{aligned}\|B^\eta_\pm(t)-B_\pm(t)\|_{k,\gamma}&\leq\Big\|\mathbb{E}\sum_{i=1,2,3}\big[\nabla(A^i_{\pm0}\circ\xi^\eta_{\pm t,0})\times\nabla\xi^{\eta i}_{\pm t,0}-\nabla(A^i_{\pm0}\circ\xi_{\pm t,0})\times\nabla\xi^i_{\pm t,0}\big]\Big\|_{k,\gamma}\\&\leq\mathbb{E}\Big\|\sum_{i=1,2,3}\big[\nabla(A^i_{\pm0}\circ\xi^\eta_{\pm t,0})\times\nabla\xi^{\eta i}_{\pm t,0}-\nabla(A^i_{\pm0}\circ\xi_{\pm t,0})\times\nabla\xi^i_{\pm t,0}\big]\Big\|_{k,\gamma}.\end{aligned}$$

Using this estimate and following the proof of the contraction property of the map $\mathcal{M}$, done in the proof of Theorem 3, we can show that there exists a constant $C_\star=C_\star(k,\gamma,d_e,d_i)$ such that

$$\|B^\eta_\pm(t)-B_\pm(t)\|_{k,\gamma}\leq C_\star\frac{V}{L^2}\mathbb{E}\|\xi^\eta_{\pm t,0}-\xi_{\pm t,0}\|_{k+1,\gamma}.$$

Combining this estimate with estimate (118) in which we take $s=0$, we obtain

$$\|B^\eta_\pm(t)-B_\pm(t)\|_{k,\gamma}\leq C_\sharp\frac{V}{L^2}\Big(\sqrt{\eta_\pm t}\;+\;\int_0^t \mathrm{d}r\,\|v^\eta_\pm(r)-v_\pm(r)\|_{k+1,\gamma}\Big)\exp\Big(C_\sharp\frac{V}{L}t\Big).\tag{119}$$

Using the continuity of the linear operator $\mathcal{F}=(\mathcal{F}_+,\mathcal{F}_-)$ from $\mathscr{C}^{k,\gamma}(\mathbb{T}^3_L;\mathbb{R}^3)\times\mathscr{C}^{k,\gamma}(\mathbb{T}^3_L;\mathbb{R}^3)$ into $\mathscr{C}^{k+1,\gamma}(\mathbb{T}^3_L;\mathbb{R}^3)\times\mathscr{C}^{k+1,\gamma}(\mathbb{T}^3_L;\mathbb{R}^3)$ with the continuity constant $C_{\mathcal{F}}$, we obtain

$$\begin{aligned}\|v^\eta_\pm(r)-v_\pm(r)\|_{k+1,\gamma}&=\|\mathcal{F}_\pm(B^\eta_+(r),B^\eta_-(r))-\mathcal{F}_\pm(B_+(r),B_-(r))\|_{k+1,\gamma}\\&=\|\mathcal{F}_\pm(B^\eta_+(r)-B_+(r),B^\eta_-(r)-B_-(r))\|_{k+1,\gamma}\\&\leq C_{\mathcal{F}}L\big(\|B^\eta_+(r)-B_+(r)\|_{k,\gamma}+\|B^\eta_-(r)-B_-(r)\|_{k,\gamma}\big).\end{aligned}\tag{120}$$

We set $X(t):=\|B^\eta_+(t)-B_+(t)\|_{k,\gamma}+\|B^\eta_-(t)-B_-(t)\|_{k,\gamma}$. Using estimates (119)–(120) and a Gronwall lemma, we obtain

$$\begin{aligned}X(t)&\leq\;C_\sharp\frac{V}{L^2}\Big(\sqrt{(\eta_++\eta_-)t}\;+\;L\int_0^t\mathrm{d}r\,X(r)\Big)\exp\Big(C_\sharp\frac{V}{L}t\Big)\\&\leq\;C_\sharp\frac{V}{L^2}\sqrt{(\eta_++\eta_-)t}\exp\Big(C_\sharp\frac{VT}{L}\Big)\exp\Big(C_\sharp\frac{VT}{L}\exp\Big(C_\sharp\frac{VT}{L}\Big)\Big)\\&\leq\;C_\sharp\frac{V}{L^2}\sqrt{(\eta_++\eta_-)t}\exp(C_\sharp\delta)\exp\big(C_\sharp\delta\exp(C_\sharp\delta)\big)\\&\leq\;C_\sharp\frac{V}{L^2}\sqrt{(\eta_++\eta_-)t}\leq\;C_\sharp\frac{V}{L^2}\sqrt{|\eta|t},\end{aligned}$$

which ends the proof of the first estimate of Theorem 4. Combining this last estimate with estimate (120), we obtain the second estimate of Theorem 4, whose the proof is complete.

4.3. Global existence of classical solutions to the SL-IXMHD equations

Here, we give a proof of the global-in-time existence of the SL-IXMHD Eqs. (92)–(94) for large resistivity or small initial data. The precise statement of this result is the following.

Theorem 5. *Let $k \geq 2$ and $A_{\pm 0} \in \mathscr{C}^{k+1,\gamma}(\mathbb{T}^3_L;\mathbb{R}^3)$ be divergence free. Let $\mathcal{R}^{\pm}_m := L(\|A_{+0}\|_{k+1,\gamma} + \|A_{-0}\|_{k+1,\gamma})/\eta_\pm$ be the magnetic Reynolds numbers of the flow. Then, there exist a time*

$$T = T\big(k,\gamma,d_e,d_i,L/\{\|A_{+0}\|_{k+1,\gamma} + \|A_{-0}\|_{k+1,\gamma}\}\big)$$

and a constant $\mathcal{R}^*_m = \mathcal{R}^*_m(k,\gamma,d_e,d_i)$ such that for all $0 < \mathcal{R}^{\pm}_m < \mathcal{R}^*_m$, the solution $(\xi_\pm, v_\pm, B_\pm)$ of the SL-IXMHD Eqs. (92)–(94) with magnetic Reynolds numbers $\mathcal{R}^{\pm}_m$ and initial data $A_{\pm 0}$ is in $\mathscr{C}([0,T]\times[0,T];\mathscr{C}^{k+1,\gamma}(\mathbb{T}^3_L;\mathbb{T}^3_L)) \times \mathscr{C}([0,T];\mathscr{C}^{k+1,\gamma}(\mathbb{T}^3_L;\mathbb{R}^3)) \times \mathscr{C}([0,T];\mathscr{C}^{k,\gamma}(\mathbb{T}^3_L;\mathbb{R}^3))$ and satisfies

$$\|B_\pm(T)\|_{k,\gamma} \leq \|B_{\pm 0}\|_{k,\gamma},$$

with $B_{\pm 0} = \nabla \times A_{\pm 0}$.

Remark 13. Theorem 5 says that for magnetic Reynolds numbers $\mathcal{R}^{\pm}_m$ small enough we have the global-in-time existence of classical solutions for the SL-IXMHD Eqs. (92)–(94). This statement contains two cases. If the resistivity parameters $\eta_\pm$ are fixed, then $\mathcal{R}^{\pm}_m$ small means $\|A_{\pm 0}\|_{k+1,\gamma}$ small; this is the case of small initial data. If the norm of the initial data $\|A_{\pm 0}\|_{k+1,\gamma}$ are fixed, then $\mathcal{R}^{\pm}_m$ small means $\eta_\pm$ large; this is the case of large resistivity parameters.

Proof of Theorem 5. The proof is divided in three steps. The first step consists in obtaining some basic a priori estimates for the magnetic fields $B_\pm$ by using our new Lagrangian formulation. The second step is devoted to obtain some Bismut-type estimates in the same spirit as in [168]. Finally, the third step combines the two previous steps with an inductive argument to show the estimate $\|B_\pm(T)\|_{k,\gamma} \leq \|B_{\pm 0}\|_{k,\gamma}$.

Step 1. We denote by ε_{lmn} the standard Levi–Civita symbol, which gives the sign of the permutation of (l,m,n). The symbol δ_{ij} is the standard Kronecker symbol. Here, we use the Einstein summation convention, which stipulates that an index appearing twice in a single term implies summation of that term over all the values of the index. Using our Lagrangian formulation, we have for $k \geq 2$ and $\ell \in \{1,2,3\}$,

$$\|B^\ell_\pm(t)\|_{k,\gamma} = \|\mathbb{E}[\varepsilon_{\ell mn}\partial_j A^i_{\pm 0}\circ\xi_{\pm t,0}\,\partial_m\xi^j_{\pm t,0}\partial_n\xi^i_{\pm t,0}]\|_{k,\gamma}. \tag{121}$$

Using $B^\ell_{\pm 0} = \varepsilon_{\ell mn}\partial_m A^n_{\pm 0}$, we observe the following decomposition,

$$\begin{aligned}\varepsilon_{\ell mn}\partial_j A^i_{\pm 0}\circ\xi_{\pm t,0}\,\partial_m\xi^j_{\pm t,0}\partial_n\xi^i_{\pm t,0} = \;& B^\ell_{\pm 0}\circ\xi_{\pm t,0}\\ &+\; \varepsilon_{\ell mn}\partial_j A^i_{\pm 0}\circ\xi_{\pm t,0}(\partial_n\xi^i_{\pm t,0}-\delta_{ni})\delta_{mj}\\ &+\; \varepsilon_{\ell mn}\partial_j A^i_{\pm 0}\circ\xi_{\pm t,0}(\partial_m\xi^j_{\pm t,0}-\delta_{mj})\delta_{ni}\\ &+\; \varepsilon_{\ell mn}\partial_j A^i_{\pm 0}\circ\xi_{\pm t,0}(\partial_n\xi^i_{\pm t,0}-\delta_{ni})(\partial_m\xi^j_{\pm t,0}-\delta_{mj}).\end{aligned}$$

From this decomposition and using the algebra property of the space $\mathscr{C}^{k,\gamma}$, we obtain for $\ell \in \{1,2,3\}$,

$$\begin{aligned}\|B^\ell_\pm(t)\|_{k,\gamma} \leq\;& \|\mathbb{E}[B^\ell_{\pm 0}\circ\xi_{\pm t,0}]\|_{k,\gamma}\\ &+\; 2C_a\mathbb{E}\big[\|DA_{\pm 0}\circ\xi_{\pm t,0}\|_{k,\gamma}\|D\varphi_{\pm t,0}\|_{k,\gamma}\big]\\ &+\; C_a\mathbb{E}\big[\|DA_{\pm 0}\circ\xi_{\pm t,0}\|_{k,\gamma}\|D\varphi_{\pm t,0}\|^2_{k,\gamma}\big].\end{aligned} \tag{122}$$

From Theorem 3 and Lemma 2, there exists a constant $C_\flat\big(k,\gamma,d_e,d_i,VT/L < \delta(k,\gamma,d_e,d_i)\big) = C_\flat(k,\gamma,d_e,d_i)$ such that

$$\|D\varphi_{\pm t,0}\|_{k,\gamma} \le C_\flat \frac{Vt}{L}, \tag{123}$$

$$\|DA_{\pm 0}\circ\xi_{\pm t,0}\|_{k,\gamma} \le \|DA_{\pm 0}\|_{k,\gamma}(1+\|\xi_{\pm t,0}\|_{k-1,\gamma})^{k+\gamma} \le \frac{C_\flat}{L}\|A_{\pm 0}\|_{k+1,\gamma}. \tag{124}$$

Using these estimates in (122), we obtain for $k \ge 2$ and $\ell \in \{1,2,3\}$,

$$\begin{aligned}\|B^\ell_\pm(t)\|_{k,\gamma} &\le \|\mathbb{E}[B^\ell_{\pm 0}\circ\xi_{\pm t,0}]\|_{k,\gamma} \ + \ C_\flat\frac{Vt}{L^2}\left(1+\frac{Vt}{L}\right)\|A_{\pm 0}\|_{k+1,\gamma}\\ &\le \|\mathbb{E}[B^\ell_{\pm 0}\circ\xi_{\pm t,0}]\|_{k,\gamma} \ + \ C_\flat\frac{Vt}{L^2}(1+\delta)\|A_{\pm 0}\|_{k+1,\gamma}\\ &\le \|\mathbb{E}[B^\ell_{\pm 0}\circ\xi_{\pm t,0}]\|_{k,\gamma} \ + \ C_\flat\frac{Vt}{L^2}\|A_{\pm 0}\|_{k+1,\gamma}.\end{aligned} \tag{125}$$

Step 2. In order to deal with the terms $\mathbb{E}[B^\ell_{\pm 0}\circ\xi_{\pm t,0}]$ in (125), we use the following Bismut formula [27]. Let $f \in \mathscr{C}^1(\mathbb{T}^3_L;\mathbb{R})$ and let $\xi_{\pm t,s}(x)$ be the solution of the backward stochastic differential Eq. (98). Then, the Bismut formula reads

$$D\mathbb{E}f(\xi_{\pm t,0})(x) = \frac{1}{t\sqrt{2\eta_\pm}}\mathbb{E}\left[f(\xi_{\pm t,0}(x))\int_t^0 (rD^T v_\pm(r)\circ\xi_{\pm t,r}(x) - I)\mathrm{d}W(r)\right]. \tag{126}$$

We now show that there exists a constant $C_\flat = C_\flat(k,\gamma,d_e,d_i)$ such that

$$|\mathbb{E}f\circ\xi_{\pm t,0}|_1 \le C_\flat\frac{L}{\sqrt{\eta_\pm t}}\|f\|_0, \tag{127}$$

$$|\mathbb{E}f\circ\xi_{\pm t,0}|_{1,\gamma} \le C_\flat\left(V\sqrt{\frac{t}{\eta_\pm}} + \frac{L}{\sqrt{\eta_\pm t}}\right)\|f\|_{0,\gamma}. \tag{128}$$

We start by the proof of (127). Using the Bismut formula (126), the Cauchy–Schwarz inequality, the Itô's isometry and Theorem 3, we obtain

$$\begin{aligned}|D\mathbb{E}f(\xi_{\pm t,0}(x))| &\le \frac{1}{t\sqrt{2\eta_\pm}}\mathbb{E}\left[|f(\xi_{\pm t,0}(x))|\left|\int_0^t (rD^T v_\pm(r)\circ\xi_{\pm t,r}(x) - I)\mathrm{d}W(r)\right|\right]\\ &\le \frac{1}{t\sqrt{2\eta_\pm}}\Big(\mathbb{E}\big[|f(\xi_{\pm t,0}(x))|^2\big]\Big)^{1/2}\left(\mathbb{E}\left[\left|\int_0^t (rD^T v_\pm(r)\circ\xi_{\pm t,r}(x) - I)\mathrm{d}W(r)\right|^2\right]\right)^{1/2}\\ &\le \frac{1}{t\sqrt{2\eta_\pm}}\sup_\Omega\|f\circ\xi_{\pm t,0}\|_{L^\infty(\mathbb{T}^3_L)}\left(\int_0^t \mathrm{d}r\,\mathbb{E}\big[|rD^T v_\pm(r)\circ\xi_{\pm t,r}(x) - I|^2\big]\right)^{1/2}\\ &\le \frac{1}{t\sqrt{\eta_\pm}}\sup_\Omega\|f\circ\xi_{\pm t,0}\|_{L^\infty(\mathbb{T}^3_L)}\Big(t + t^3\|Dv_\pm\|^2_{L^\infty([0,T]\times\mathbb{T}^3_L)}\Big)^{1/2}\\ &\le \frac{1}{\sqrt{\eta_\pm t}}\|f\|_{L^\infty(\mathbb{T}^3_L)}\Big(1 + t\|Dv_\pm\|_{L^\infty([0,T]\times\mathbb{T}^3_L)}\Big)\\ &\le \frac{1}{\sqrt{\eta_\pm t}}\|f\|_{L^\infty(\mathbb{T}^3_L)}\left(1+\frac{Vt}{L}\right)\\ &\le \frac{C_\flat}{\sqrt{\eta_\pm t}}\|f\|_0.\end{aligned}$$

From this estimate and the definition of the non-dimensional norm $|\cdot|_1$, we deduce directly estimate (127). We continue with the proof of (128). Using the Bismut formula (126), we obtain

$$L^{1+\gamma}|D\mathbb{E}f(\xi_{\pm t,0}(x)) - D\mathbb{E}f(\xi_{\pm t,0}(y))|\,|x-y|^{-\gamma} = \frac{L^{1+\gamma}}{t\sqrt{2\eta_\pm}}|x-y|^{-\gamma}$$

$$
\mathbb{E}\bigg[\big\{f(\xi_{\pm t,0}(x))-f(\xi_{\pm t,0}(y))\big\}\int_t^0(rD^Tv_\pm(r)\circ\xi_{\pm t,r}(x)-I)\mathrm{d}W(r)
+f(\xi_{\pm t,0}(y))\int_t^0 r\big(D^Tv_\pm(r)\circ\xi_{\pm t,r}(x)-D^Tv_\pm(r)\circ\xi_{\pm t,r}(y)\big)\mathrm{d}W(r)\bigg]. \tag{129}
$$

Using the Cauchy–Schwarz inequality, the Itô's isometry and Theorem 3, we obtain from (129),

$$
\begin{aligned}
&L^{1+\gamma}|D\mathbb{E}f(\xi_{\pm t,0}(x))-D\mathbb{E}f(\xi_{\pm t,0}(y))|\,|x-y|^{-\gamma}\\
&\le \frac{L^{1+\gamma}}{t\sqrt{2\eta_\pm}}|x-y|^{-\gamma}\Bigg\{\Big(\mathbb{E}\big[|f(\xi_{\pm t,0}(x))-f(\xi_{\pm t,0}(y))|^2\big]\Big)^{1/2}\\
&\Big(\mathbb{E}\Big[\Big|\int_0^t(rD^Tv_\pm(r)\circ\xi_{\pm t,r}(x)-I)\mathrm{d}W(r)\Big|^2\Big]\Big)^{1/2}+\Big(\mathbb{E}\big[|f(\xi_{\pm t,0}(y))|^2\big]\Big)^{1/2}\\
&\Big(\mathbb{E}\Big[\Big|\int_0^t r\big(D^Tv_\pm(r)\circ\xi_{\pm t,r}(x)-D^Tv_\pm(r)\circ\xi_{\pm t,r}(y)\big)\mathrm{d}W(r)\Big|^2\Big]\Big)^{1/2}\Bigg\}\\
&\le \frac{1}{t\sqrt{\eta_\pm}}\sup_{\Omega}\Bigg\{L|f\circ\xi_{\pm t,0}|_\gamma\Big(t+t^3\|Dv_\pm\|^2_{L^\infty([0,T]\times\mathbb{T}^3_L)}\Big)^{1/2}\\
&\quad+\|f\circ\xi_{\pm t,0}\|_0\Big(t^3L^2\big|Dv_\pm\circ\xi_\pm\big|^2_{L^\infty([0,T]\times[0,T];\mathscr{C}^\gamma(\mathbb{T}^3_L))}\Big)^{1/2}\Bigg\}\\
&\le \frac{1}{t\sqrt{\eta_\pm}}\sup_{\Omega}\Bigg\{|f|_\gamma L\|D\xi_{\pm t,0}\|_0^\gamma t^{1/2}\Big(1+C_\flat\frac{Vt}{L}\Big)\\
&\quad+\|f\|_0t^{3/2}L\big|Dv_\pm\big|_{L^\infty([0,T];\mathscr{C}^\gamma(\mathbb{T}^3_L))}\|D\xi_{\pm t,0}\|^\gamma_{L^\infty([0,T]\times[0,T]\times\mathbb{T}^3_L)}\Bigg\}\\
&\le \frac{1}{t\sqrt{\eta_\pm}}\sup_{\Omega}\Big\{C_\flat|f|_\gamma Lt^{1/2}+C_\flat|f|_\gamma Vt^{3/2}+C_\flat\|f\|_0Vt^{3/2}\Big\}\\
&\le C_\flat\left(V\sqrt{\frac{t}{\eta_\pm}}+\frac{L}{\sqrt{\eta_\pm t}}\right)\|f\|_{0,\gamma},
\end{aligned}
$$

which ends the proof of (128).

Step 3. Here, for $k\ge 2$ and $\ell\in\{1,2,3\}$, we estimate the term $\|\mathbb{E}[B^\ell_{\pm0}\circ\xi_{\pm t,0}]\|_{k,\gamma}$, appearing in (125). We start with $k=0$. Using (123)–(124) and (127), we obtain

$$
\begin{aligned}
\|\mathbb{E}[B^\ell_{\pm0}\circ\xi_{\pm t,0}]\|_0 &= \|\mathbb{E}[(\nabla\times(A_{\pm0}\circ\xi_{\pm t,0}))^\ell+(\nabla\times A_{\pm0}\circ\xi_{\pm t,0})^\ell-(\nabla\times(A_{\pm0}\circ\xi_{\pm t,0}))^\ell]\|_0\\
&\le \|\nabla\times\mathbb{E}[A_{\pm0}\circ\xi_{\pm t,0}]\|_0+\sup_\ell\mathbb{E}\big[\|\varepsilon_{\ell mn}\partial_iA^n_{\pm0}\circ\xi_{\pm t,0}(\delta_{mi}-\partial_m\xi^i_{\pm t,0})\|_0\big]\\
&\le \|D\mathbb{E}[A_{\pm0}\circ\xi_{\pm t,0}]\|_0+\mathbb{E}\big[\|DA_{\pm0}\circ\xi_{\pm t,0}(I-D\xi_{\pm t,0})\|_0\big]\\
&\le \|D\mathbb{E}[A_{\pm0}\circ\xi_{\pm t,0}]\|_0+\mathbb{E}\big[\|DA_{\pm0}\circ\xi_{\pm t,0}D\varphi_{\pm t,0}\|_0\big]\\
&\le \|D\mathbb{E}[A_{\pm0}\circ\xi_{\pm t,0}]\|_0+\mathbb{E}\big[\|DA_{\pm0}\circ\xi_{\pm t,0}\|_0\|D\varphi_{\pm t,0}\|_0\big]\\
&\le \frac{C_\flat}{\sqrt{\eta_\pm t}}\|A_{\pm0}\|_0+C_\flat\frac{Vt}{L^2}|A_{\pm0}|_1\\
&\le C_\flat\Big(\frac{1}{\sqrt{\eta_\pm t}}+\frac{Vt}{L^2}\Big)\|A_{\pm0}\|_1,
\end{aligned} \tag{130}
$$

which completes the treatment of the case $k=0$. For $k=1$, we have to bound only the semi-norm $|\mathbb{E}[B^\ell_{\pm0}\circ\xi_{\pm t,0}]|_1$. Using (127) we obtain

$$
|\mathbb{E}[B^\ell_{\pm0}\circ\xi_{\pm t,0}]|_1\le\frac{C_\flat L}{\sqrt{\eta_\pm t}}\|B_{\pm0}\|_0\le\frac{C_\flat}{\sqrt{\eta_\pm t}}\|A_{\pm0}\|_1. \tag{131}
$$

For $k = 2$, we have to estimate $|\mathbb{E}[B^\ell_{\pm 0} \circ \xi_{\pm t,0}]|_2$ and $|\mathbb{E}[B^\ell_{\pm 0} \circ \xi_{\pm t,0}]|_{2,\gamma}$. We start with $|\mathbb{E}[B^\ell_{\pm 0} \circ \xi_{\pm t,0}]|_2$. Observe the following decomposition,

$$D^2\mathbb{E}[B^\ell_{\pm 0} \circ \xi_{\pm t,0}] = D\mathbb{E}[DB^\ell_{\pm 0} \circ \xi_{\pm t,0}] \ + \ D\mathbb{E}[DB^\ell_{\pm 0} \circ \xi_{\pm t,0}(D\xi_{\pm t,0} - I)]. \tag{132}$$

Using (127) for the first term of the right-hand side of (132) and using Lemma 2 for the second term of the right-hand side of (132), we obtain

$$\begin{aligned} |\mathbb{E}[B^\ell_{\pm 0} \circ \xi_{\pm t,0}]|_2 &\le \frac{C_\flat L^2}{\sqrt{\eta_\pm t}} \|DB^\ell_{\pm 0}\|_0 \ + \ \frac{C_\flat V t}{L} L^2 \|D^2 B^\ell_{\pm 0}\|_0 \\ &\le \frac{C_\flat}{\sqrt{\eta_\pm t}} \|A_{\pm 0}\|_2 \ + \ \frac{C_\flat V t}{L^2} \|A_{\pm 0}\|_3. \end{aligned} \tag{133}$$

We continue with $|\mathbb{E}[B^\ell_{\pm 0} \circ \xi_{\pm t,0}]|_{2,\gamma}$. Using (128) for the first term of the right-hand side of (132) and using Lemma 2 for the second term of the right-hand side of (132), we obtain

$$\begin{aligned} |\mathbb{E}[B^\ell_{\pm 0} \circ \xi_{\pm t,0}]|_{2,\gamma} &\le C_\flat L\left(V\sqrt{\frac{t}{\eta_\pm}} \ + \ \frac{L}{\sqrt{\eta_\pm t}}\right) \|DB^\ell_{\pm 0}\|_{0,\gamma} \ + \ \frac{C_\flat V t}{L} L^2 \|D^2 B^\ell_{\pm 0}\|_{0,\gamma} \\ &\le C_\flat \left(V\sqrt{\frac{t}{\eta_\pm}} \ + \ \frac{L}{\sqrt{\eta_\pm t}}\right) \frac{\|A_{\pm 0}\|_{2,\gamma}}{L} \ + \ \frac{C_\flat V t}{L^2} \|A_{\pm 0}\|_{3,\gamma}. \end{aligned} \tag{134}$$

Gathering (130), (131), (133) and (134), we obtain

$$\|\mathbb{E}[B^\ell_{\pm 0} \circ \xi_{\pm t,0}]\|_{2,\gamma} \le C_\flat \left(V\sqrt{\frac{t}{\eta_\pm}} \ + \ \frac{L}{\sqrt{\eta_\pm t}}\right) \frac{\|A_{\pm 0}\|_{2,\gamma}}{L} \ + \ C_\flat \frac{Vt}{L} \frac{\|A_{\pm 0}\|_{3,\gamma}}{L}.$$

By induction on k, with $k \ge 2$, we obtain

$$\|\mathbb{E}[B^\ell_{\pm 0} \circ \xi_{\pm t,0}]\|_{k,\gamma} \le C_\flat \left(V\sqrt{\frac{t}{\eta_\pm}} \ + \ \frac{L}{\sqrt{\eta_\pm t}}\right) \frac{\|A_{\pm 0}\|_{k,\gamma}}{L} \ + \ C_\flat \frac{Vt}{L} \frac{\|A_{\pm 0}\|_{k+1,\gamma}}{L},$$

for $\ell \in \{1, 2, 3\}$. Using this last estimate and (125), we obtain

$$\begin{aligned} \|B^\ell_\pm\|_{k,\gamma} &\le \|\mathbb{E}[B^\ell_{\pm 0} \circ \xi_{\pm t,0}]\|_{k,\gamma} \ + \ C_\flat \frac{Vt}{L} \frac{\|A_{\pm 0}\|_{k+1,\gamma}}{L} \\ &\le C_\flat \left(V\sqrt{\frac{t}{\eta_\pm}} \ + \ \frac{L}{\sqrt{\eta_\pm t}}\right) \frac{\|A_{\pm 0}\|_{k,\gamma}}{L} \ + \ C_\flat \frac{Vt}{L} \frac{\|A_{\pm 0}\|_{k+1,\gamma}}{L}, \\ &\le C_\flat \left(V\sqrt{\frac{t}{\eta_\pm}} \ + \ \frac{L}{\sqrt{\eta_\pm t}} \ + \ \frac{Vt}{L}\right) \|B_{\pm 0}\|_{k,\gamma}. \end{aligned} \tag{135}$$

Setting

$$\mathcal{R}^\pm_m := \frac{L}{\eta_\pm}(\|A_{+0}\|_{k+1,\gamma} \ + \ \|A_{-0}\|_{k+1,\gamma}) = \frac{LV}{\kappa \eta_\pm},$$

we obtain from (135),

$$\|B^\ell_\pm\|_{k,\gamma} \le C_\flat \left(\sqrt{\mathcal{R}^\pm_m}\left(\frac{Vt}{L}\right)^{1/2} \ + \ \sqrt{\mathcal{R}^\pm_m}\left(\frac{Vt}{L}\right)^{-1/2} \ + \ \frac{Vt}{L}\right) \|B_{\pm 0}\|_{k,\gamma}. \tag{136}$$

From Theorem 3, we have for all $t \in [0, T]$, $Vt/L < 1$. Hence $(Vt/L)^{1/2} < (Vt/L)^{-1/2}$, and (136) becomes

$$\|B^\ell_\pm\|_{k,\gamma} \le C_\flat \left(\sqrt{\mathcal{R}^\pm_m}\left(\frac{Vt}{L}\right)^{-1/2} \ + \ \frac{Vt}{L}\right) \|B_{\pm 0}\|_{k,\gamma}. \tag{137}$$

Maximizing the right-hand side of (137) with respect to t, we obtain

$$T_{\pm 0} = \frac{L}{V} \sqrt[3]{\mathcal{R}^\pm_m},$$

where $T_{\pm 0}$ is the value of t for which the maximum of the right-hand side of (137) is reached. Let us define $\mathcal{R}_m := \max\{\mathcal{R}_m^+, \mathcal{R}_m^-\}$. Then $T_{\pm 0} \le T_0 := L\mathcal{R}_m^{1/3}/V$. We can choose T_0 small enough so that $T_0 < T$, where T is the local existence time of Theorem 3. Indeed, it is sufficient that $\mathcal{R}_m^{\pm} \le \mathcal{R}_m < (VT/L)^3 \le \delta^3$. For the time T_0 we obtain

$$\|B_{\pm}^{\ell}(T_0)\|_{k,\gamma} \le C_\flat \mathcal{R}_m^{1/3} \|B_{\pm 0}\|_{k,\gamma} \le \|B_{\pm 0}\|_{k,\gamma},$$

for $\mathcal{R}_m$ small enough, i.e., $\mathcal{R}_m^{1/3} \le C_\flat^{-1}$. Taking $\mathcal{R}_m^* := \min\{\delta^3, C_\flat^{-3}\} = \mathcal{R}_m^*(k, \gamma, d_e, d_i)$, there exists a time $T_0 \le T$ such that for all $\mathcal{R}_m^{\pm} \le \mathcal{R}_m < \mathcal{R}_m^*$, we obtain

$$\|B_{\pm}^{\ell}(T_0)\|_{k,\gamma} \le \|B_{\pm 0}\|_{k,\gamma}.$$

The above proof, which holds on the time interval $[0, T_0]$, can be repeated on the time interval $[T_0, T_1]$ with $T_1 > T_0$ by taking $B_{\pm}(T_0)$ or $A_{\pm}(T_0)$ as new initial conditions, and so on we obtain a global-in-time classical solution. □

Data availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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